UNDER THE HOOD OF A CARBON FOOTPRINT CALCULATOR

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ABSTRACT. We explain the basic linear algebra formulas used in carbon footprints computations.

Have you computed your carbon footprint lately? Wondered where all those numbers came from? Two main methods for calculating carbon footprints are the *Input-Output* method and the *Lifecycle Analysis* method, also known as *Craddle-to-Grave*. Both aim to quantify the emissions of a product or service, but the methods (and reliability) are fundamentally different.

Input-Output is an economic approach developed in [Leo], widely accepted since the 60ies. It uses national or regional data on industries and their interdependencies. It relies on aggregated data about economic sectors (e.g., agriculture, manufacturing) and their emissions, and we shall see in this note (Definition 2) the formula estimating emissions associated with production, supply chains, and services. This is the commonly accepted method for large-scale, economy-wide assessments, is not data-intensive, relies on publicly available economic data. It will lack detail on specific products and miss the emissions from smaller, niche activities, but offers the warranty that all emissions are taken into account (Proposition 1).

Lifecycle Analysis estimates a specific product's environmental impact over its entire lifecycle, from raw material extraction to disposal. However, the under or over counting of secondary emissions are only estimated and the total complexity makes a consistency check too data-intensive and time-consuming to actually carry out.

A concrete question. To estimate the carbon footprint of a T-shirt for instance, one needs the footprint of the cotton all the way from production of the raw material up to the stitching of the T-shirt, of the washing during its lifespan and of the disposal of the unusable T-shirt. But to estimate the cotton footprint only, one needs the transport of the water, the production of the pesticides to protect the crop, the

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fertilizers, all those requiring tractors and fuel, the fuel itself requiring transport and fuel, thus creating loops that are difficult to extricate. And once estimated all the emissions, how do we know that all the emissions of all the objects that we so compute, actually sum up to all the emissions that we actually produce? The Input-Output method takes the problem from the other end: it takes *all* the measured emissions, and then attributes it to different sectors, and then computes the carbon footprint of a goods from the textile industry according to its direct emissions, combined with all the indirect emissions as we now explain.

Basic Input-Output. This method models a closed economy with n *economical sectors*, $S = (S_1, \ldots, S_n)$, and comes in the following form:

	S_1	S_2	$\ldots$	S_n	D	T
S_1	c_{11}	c_{12}	$\ldots$	c_{1n}	d_1	t_1
S_2	c_{21}	c_{22}	$\ldots$	c_{2n}	d_2	t_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
S_n	c_{n1}	c_{n2}	$\ldots$	c_{nn}	d_n	t_n
V	v_1	v_2	$\ldots$	v_n		
T	t_1	t_2	$\ldots$	t_n		

The $n \times n$ *transaction matrix* is $C = (c_{ij}) \in M_n(\mathbf{R}_+)$ (positive coefficients only) has entry c_{ij} the number representing the economical transactions from sector S_i to S_j. The diagonal entries c_{ii} then represent the internal gross revenue of the sector. The *demand vector* $D = (d_1, \ldots, d_n)$ expresses the money spent by the population on each sector: d_i is the amount bought from sector S_i, for $i = 1, \ldots, n$. The *added value* vector is given by $V = (v_1, \ldots, v_n)$ and the *total* vector $T = (t_1, \ldots, t_n)$ satisfy the following relations:

$$t_i = \sum_{j=1}^{n} c_{ij} + d_i = v_i + \sum_{j=1}^{n} c_{ji}$$

for $i = 1, \ldots, n$. The value t_i represents the *total output* of the sector S_i, which is also the *total input* because the system is a closed circuit.

Definition 1. The *technical coefficient matrix* $A = (a_{ij})$ is given by

$$a_{ij} = \frac{c_{ij}}{t_j},$$

so that $A = C D_T^{-1}$ where D_T denotes the diagonal matrix with entries $t_1, \ldots, t_n$ in the diagonal.

Recall that the scalar product of two vectors $X = (xi, \ldots, x_n)$ and $Y^t = (y_1, \ldots, y_n)$ is given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = XY$$

where X is a *line vectors*, namely $1 \times n$ matrix, and Y is a *column vector*, namely an $n \times 1$ matrix and XY is the usual matrix multiplication so a real number.

Footprints. Economical activities have all sorts of footprints that are correlated to the income and the cost of things. To compute the footprint we need to have a *emission* vector $E = (e_1, \ldots, e_n)$ where e_i is the direct emission of sector S_i. It could be millions of tons of CO_2, measured or estimated directly above the industries of the sector, or square feet measured by the occupancy of the industries, or tons of plastic trash estimated by the collectivities, etc. Then $|E| = e_1 + \cdots + e_n$ is the total footprint of the economical system. The *direct intensity* is given by $F = (f_1, \ldots, f_n)$ with

$$f_i = \frac{e_i}{t_i}$$

each entry is the print per unit of money. Now, if we compute the inner product

$$\langle D, F \rangle = \sum_{i=1}^{n} d_i f_i$$

it is the footprint directly attributed to the consumers, but it doesn't take into account all the intermediate exchanges that the sectors had to create the product. If you buy goods for k dollars to industry S_i, then $k f_i$ indicates the direct footprint of the purchase, but not the total footprint of the goods. That one has to take into account the exchanges the sector S_i had from the other ones.

Definition 2. The *total intensity vector* $X = (x_1, \cdots, x_n)$ giving the total footprint of the sector S_i by money amount of final consumer demand, is given by

$$X = F + FA + FA^2 + \cdots = F(I - A)^{-1}$$

where I denotes the $n \times n$ identity matrix and A the technical coefficients matrix from Definition 1.

Notice that since most matrices are invertible we'll assume that this one is.

Proposition 1. *The total footprint is completely attributed to the consumers. Namely, with the above notations, $\langle X, D \rangle = |E|$.*

Proof. First notice that the equations $t_i = \sum_{j=1}^{n} c_{ij} + d_i$ for $i = 1, \ldots, n$ rephrase as

$$D = T - C\mathbf{1} = T - AD_T\mathbf{1} = T - AT = (I - A)T,$$

where $\mathbf{1} = (1, \ldots, 1)$ and T is the total vector and D the demand vector, both viewed as column vectors. We compute:

$$\langle X, D \rangle = \left\langle F(I - A)^{-1}, (I - A)T \right\rangle = FT = \sum_{i=1}^{n} \frac{e_i}{t_i} t_i = |E|$$

$\square$

Waste. In a closed economy, in order to take into account waste disposal, one needs a sector S_i that runs the waste management. That sector has a large footprint since incinerators produce large quantities of CO_2 (of the order of 1kg CO_2 per kg of waste incinerated [Waste]), and is eventually attributed to the consumers through waste management taxes (in France the weight of waste is estimated at half a ton per person and per year). The model also only doesn't discuss demand and waste, assuming that everything is bought and nothing gets wasted.

Inverses. If B is a matrix ϵ-close to A (say in terms of the coefficients), then the distance between $(I - A)^{-1}$ and $(I - B)^{-1}$ could be a priori be as big as we want, for instance if the matrices both have an eigenvalue close enough to 1. Since all the values are estimated, the model is probably not very stable in an optimization exercise.

Added value. The consumer footprint doesn't take into account the added value V, which is what the companies make, and the model attributes the footprint solely to the consumers, through the vector D. We could define *the total systemic vector*

$$Y = F(I - A^t)^{-1}$$

and check that $\langle Y, V \rangle = |E|$. That vector distributes the total footprint to the added value of the different sectors. That attribution seems more natural than the consumer one, as it links the footprint to the benefits of the companies, see also the 2021 opinion piece [NYTop].

Concretely. The tables are quite available, see for Switzerland [IOCH], but a measure of environmental impact for each sector appearing in the table is harder to obtain, especially as it needs to be computed/collected in an homogeneous manner over the different sectors in order to make sense.

References

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