Distortion in connected Lie groups and bounded cohomology

Indira Chatterji

(joint work with Guido Mislin, Christophe Pittet, Laurent Saloff-Coste)

Given a locally compact and compactly generated group G and a $\mathbf{Z}$-subgroup of G, generated by an element $g \in G$ one can compare the length of $g^n \in G$ (for the length coming from a compact generating set of G, or a Riemannian metric on G if G happens to be a connected Lie group) with n, which is the length of g^n as an element of the $\mathbf{Z}$-subgroup. If those two lengths have the same asymptotic behavior, the $\mathbf{Z}$-subgroup generated by g is quasi-isometrically embedded. In this case we say that it is *undistorted* and we call it *distorted* otherwise. Distortion never occurs for semi-simple elements when there is enough non-positive curvature but does for parabolic ones and for Heisenberg groups. Given a locally compact group G and a central $\mathbf{Z}$-extension

$$0 \to \mathbf{Z} \to E \to G \to \{e\}$$

it is a classical result in group theory that the law on the group E is given by a 2-cocycle $\gamma : G \times G \to \mathbf{Z}$ expressing the failure of a section $s : G \to E$ to be a homomorphism. If that 2-cocycle γ is bounded, it is a straightforward computation that the inclusion $\mathbf{Z} \to E$ is undistorted. Hence a natural question is the following.

Question. *If a central $\mathbf{Z}$-extension of a group G is undistorted, what can we say about the class of the 2-cocycle defining it?*

In the case of topological extensions of connected Lie groups, we provide a complete answer to this question and more precisely the following.

Main Theorem (Chatterji, Mislin, Pittet, Saloff-Coste). *Let G be a connected Lie group. The following conditions are equivalent.*

(1) *The radical $\sqrt{G}$ of G is linear.* (Topologists might prefer to think of the equivalent condition that the closure of the commutator subgroup of $\sqrt{G}$ is simply-connected.)
(2) *The map $H^n_B b(G, \mathbf{Z}) \to H^n_B(G, \mathbf{Z})$ is surjective for all $n \geq 2$.* (In words, each Borel cohomology class of G with $\mathbf{Z}$-coefficients can be represented by a Borel bounded cocycle.)
(3) *The map $H^2_B b(G, \mathbf{Z}) \to H^2_B(G, \mathbf{Z})$ is surjective.* (In words, each Borel cohomology class of G of degree two with $\mathbf{Z}$-coefficients can be represented by a Borel bounded cocycle.)
(4) *The class in $H^2_B(G, \pi_1(G))$ defined by the universal cover of G can be represented by a Borel bounded cocycle.*
(5) *The natural inclusion $\pi_1(G) \to \tilde{G}$ of the fundamental group of G into the universal cover of G is undistorted.*

Here the radical $\sqrt{G}$ of G is the maximal connected solvable subgroup of G. The groups $H^n_B(G, \mathbf{Z})$ are Borel cohomology classes of G with $\mathbf{Z}$-coefficients. Those are, as in the classical group cohomology, classes of cocycles $\gamma : G^n \to \mathbf{Z}$ up to co-boundaries, except that they are assumed to be Borel measurable. Moore [9] showed that $H^2_B(G, \mathbf{Z})$ classify topological central $\mathbf{Z}$-extensions of G (i.e covers with $\mathbf{Z}$-fibers) and Wigner [11] that $H^n_B(G, \mathbf{Z}) \simeq H^n(BG, \mathbf{Z})$, where BG is a classifying space for G.

Example. The "toy example" one should keep in mind is the following: Take $\mathbf{H}$ be the 3-dimensional Heisenberg group and consider $\mathbf{H} \times S^1$, its center is $\mathbf{R} \times S^1$. Take the discrete central subgroup $\mathbf{Z}$ generated by $(1, t)$ with 1 generating $\mathbf{Z}$ in $\mathbf{R}$, and t of infinite order in S^1; this central subgroup of $\mathbf{H} \times S^1$ is discrete. Define $G := (\mathbf{H} \times S^1)/\mathbf{Z}$. It is a nilpotent connected Lie group with $[G, G]$ homeomorphic to $\mathbf{R}$, embedded in the maximal torus $S^1 \times S^1$ of G in a dense way (hence not closed). It follows that $\pi_1([G, G])$ is trivial but $\pi_1\left(\overline{[G, G]}\right) = \mathbf{Z}^2$.

Remarks.

- The proof given in [5] shows that one can relax the boundedness hypothesis in Conditions (2), (3) and (4), of the above theorem: assuming that the representative cocycle has sub-linear growth, leads in each case to an equivalent condition. Similarly, assuming that $\pi_1(G) \to \tilde{G}$ has sub-linear distortion is equivalent to Condition (5).
- Main Theorem assumes *integer* coefficients. Let us emphasis that this is in contrast with the case of *real* coefficients (as studied in [8], [3], [1]), where the map $H^*_{Bb}(G, \mathbf{R}) \to H^*_B(G, \mathbf{R})$ conjectured to be onto for semi-simple Lie groups with finite center [4] but is never onto for G a connected solvable Lie group for which the right hand side does not vanish for all positive degrees as the left hand side is always 0. This map is not onto in degree 3 for G the universal cover of $SL(2, \mathbf{R})$ [8], thus, in general it is not onto for semi-simple Lie groups either. However it is onto for semi-simple Lie groups with finite center in the top dimension as this is equivalent to the simplicial volume conjecture solved by Lafont-Schmidt [7].

As a corollary to our Main Theorem, we obtain the following generalization of Gromov's [6] and Bucher-Karlsson's [2] theorems.

Corollary. Let G be a virtually connected Lie group with linear radical. Each class in the image of the natural map $H^*(BG, \mathbf{R}) \to H^*(BG^\delta, \mathbf{R})$ can be represented by a cocycle whose set of values on all singular simplices of BG^δ is finite.

Steps in the proof of Main Theorem. That (4) implies (5) is a generalisation to topological groups of an easy exercise in case of discrete groups. The linearity of the radical $\sqrt{G}$ of G amounts to a trivial fundamental group of the closure of the commutator subgroup of $\sqrt{G}$. So, non-linearity of that radical gives us an element in that fundamental group, which creates a distorted copy of $\mathbf{Z}$ in the universal cover of G (this uses a crucial result proved in [10] in the nilpotent case), and that in turn amounts to the existence of a unbounded class in degree 2, hence that gives us (5) implies (1). To see that (1) implies (3) (which in turn easily implies (4)), if we assume the radical to be linear, then standard Lie group structure theory allows us to reduce to the case of semi-simple Lie groups, which we then can compute. Finally, the equivalence with Condition (2), the boundedness of $\mathbf{Z}$-valued classes in all degrees, is a consequence of the observation that the $\mathbf{R}$-cohomology of a connected Lie group is generated by 2-dimensional classes and classes which are in the image via an inflation map of primary characteristic classes of a semi-simple Lie group, which according to Bucher-Karlsson's work in [2] and [1] have continuous, bounded representatives.

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