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On Property (RD)
for
Certain Discrete Groups

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Résumé. Cette thèse explore certains aspects de la propriété dite de Décroissance Rapide, propritété (DR), qui est un phénomène relevant de la géométrie non-commutative des groupes discrets. Cette propriété a récemment connu un intense regain d'intérêt dû aux travaux de V. Lafforgue qui en font usage pour démontrer la conjecture de Baum-Connes dans certains cas. Outre qu'ils constituent une évidente motivation, les travaux de V. Lafforgue ont aussi été une source d'inspiration pour une partie du présent travail.

Notre approche est de nature géométrique : nous considérons des groupes discrets d'isométries de certains espaces symétriques, notamment l'espace associé au groupe exceptionnel $E_{6(-26)}$, ainsi que de produits quelconques d'espaces hyperboliques à la Gromov, ou encore des produits mixtes impliquant les deux types d'espaces.

Abstract. We explore in this dissertation certain aspects of the Rapid Decay property, property (RD), which is a phenomenon in the non-commutative geometry of discrete groups. Due to V. Lafforgue's work on the Baum-Connes conjecture, there has recently been a considerable interest in this property. On top of this obvious motivation, V. Lafforgue's techniques were also a source of inspiration for part of the present work.

Our approach is geometrical in nature: we consider discrete groups of isometries of certain symmetric spaces, notably the space associated to the exceptional group $E_{6(-26)}$, as well as arbitrary products of Gromov hyperbolic spaces, or mixed products of both types of spaces.

Contents

Introduction 5
- Motivations and statement of results 5
- Organization of the text 6
- Acknowledgments 6

Chapter 1. Around property (RD) 9
- Basic definitions 9
- What is property (RD) good for? 21

Chapter 2. Uniform lattices in products of rank one Lie groups 25
- Two properties for a metric space 25
- Uniform nets in products of hyperbolic spaces 29
- Coxeter groups 33

Chapter 3. Triples of points in $SL_3(\mathbf{H})$ and $E_{6(-26)}$ 35
- The case of $SL_3(\mathbf{H})$ 35
- The case of $E_{6(-26)}$ 40
- Relation with triples in $SL_3(\mathbf{C})$ 47

Chapter 4. Uniform lattices in products of SL_3's and rank one's 49
- Two properties of triples of points 49
- How we establish property (RD) 52

Flat of type A_2 endowed with a Finsler distance 55

Loose ends 57

Bibliography 59

Introduction

Motivations and statement of results

A discrete group Γ is said to have *property (RD) with respect to a length function* ℓ if there exists a polynomial P such that for any $r \in \mathbf{R}_+$ and $f \in \mathbf{C}\Gamma$ supported on elements of length shorter than r the following inequality holds:

$$\|f\|_* \leq P(r)\|f\|_2$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(\Gamma)$, and $\|f\|_2$ the usual ℓ^2 norm. Property (RD) has been first established for free groups by Haagerup in [**7**], but introduced and studied by P. Jolissaint in [**12**], who established it for polynomial growth groups and for classical hyperbolic groups. The extension to Gromov hyperbolic groups is due to P. de la Harpe in [**8**]. Providing the first examples of higher rank groups, J. Ramagge, G. Robertson and T. Steger in [**24**] proved that property (RD) holds for discrete groups acting freely on the vertices of an $\tilde{A}_1 \times \tilde{A}_1$ or $\tilde{A}_2$ building and recently V. Lafforgue did it for cocompact lattices in $SL_3(\mathbf{R})$ and $SL_3(\mathbf{C})$ in [**14**]. Cocompactness is crucial since the only (up to now) known obstruction to property (RD) has been given by P. Jolissaint in [**12**] and is the presence of an amenable subgroup with exponential growth. This has been turned into a conjecture:

Conjecture 1 (A. Valette, see [**30**] or [**2**]). *Property (RD) with respect to the word length holds for any discrete group acting isometrically, properly and cocompactly either on a Riemannian symmetric space or on an affine building.*

Property (RD) is important in the context of Baum-Connes conjecture, precisely, V. Lafforgue in [**15**] proved that for "good" groups having property (RD), the Baum-Connes conjecture without coefficients holds. The main result of this thesis is the following

Theorem 0.1. *Any discrete cocompact subgroup* Γ *of a finite product of type*

$$\mathrm{Iso}(\mathcal{X}_1) \times \cdots \times \mathrm{Iso}(\mathcal{X}_n)$$

has property (RD) with respect to the word length, where the $\mathcal{X}_i$*'s are either complete locally compact Gromov hyperbolic spaces,* $\tilde{A}_2$*-buildings, or symmetric spaces associated to* $SL_3(\mathbf{R})$*,* $SL_3(\mathbf{C})$*,* $SL_3(\mathbf{H})$ *and* $E_{6(-26)}$.

This provides many interesting examples of discrete groups having property (RD), such as cocompact lattices in $SL_2(\mathbf{R})\times SL_2(\mathbf{R})$. Notice that combining our result with V. Lafforgue's crucial theorem in [**15**] yields the following

COROLLARY 0.2. *The Baum-Connes conjecture without coefficients (see* [**30**]*) holds for any cocompact lattice in*

$$G = G_1 \times \cdots \times G_n$$

where the G_i's are either rank one Lie groups, $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ or $E_{6(-26)}$.

Simultaneously and independently, M. Talbi in his Ph.D. thesis (see [**28**]) proved that property (RD) holds for groups acting on buildings of type $\tilde{A}_{i_1} \times \cdots \times \tilde{A}_{i_k}$, where $i_j \in \{1, 2\}$.

Organization of the text

Chapter 1 of this work is a general exposition of property (RD), essentially based on P. Jolissaint's results in [**12**]. We slightly improve his result on split extensions by considering general length functions instead of the word length, but otherwise the proof is the same. In Chapter 2 we look at the particular case where the $\mathcal{X}_i$'s are locally compact Gromov hyperbolic spaces. In case where all the $\mathcal{X}_i$'s are trees, we can even drop the local finiteness condition, which allows to establish property (RD) for Coxeter groups. This remark is due to N. Higson. In Chapter 3, the study of $SL_3(\mathbf{H})$ and $E_{6(-26)}$ will allow us to answer (positively) a question posed by V. Lafforgue in [**14**], which was to know whether his Lemmas 3.5 and 3.7 are still true for the groups $SL_3(\mathbf{H})$ and $E_{6(-26)}$, whose associated symmetric spaces have also flats of type A_2. Observing that these lemmas are in fact "three points conditions" it will be enough to prove that if X denotes $SL_3(\mathbf{H})/SU_3(\mathbf{H})$ or $E_{6(-26)}/F_{4(-52)}$ then for any three points in X there exists a totally geodesic embedding of $SL_3(\mathbf{C})/SU_3(\mathbf{C})$ containing those three points. In Chapter 4 we will explain how to use the techniques used in [**24**] and [**14**] for the above described products. The "loose ends" chapter is about questions to which I haven't been able to answer.

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CHAPTER 1

Around property (RD)

Basic definitions

In this section we will give some conditions which are equivalent to property (RD) and study the stability of property (RD) under group extensions. Most of the results given in this section are either simple remarks or slights improvements of results contained in P. Jolissaint's paper [**12**].

Definition 1.1. Let Γ be a discrete group, a *length function* on Γ is a function $\ell : \Gamma \to \mathbf{R}_+$ satisfying:

- $\ell(e) = 0$, where e denotes the neutral element in Γ,
- $\ell(\gamma) = \ell(\gamma^{-1})$ for any $\gamma \in \Gamma$,
- $\ell(\gamma\mu) \leq \ell(\gamma) + \ell(\mu)$ for any $\gamma, \mu \in \Gamma$.

The function $d(\gamma, \mu) = \ell(\gamma^{-1}\mu)$ is a left Γ-invariant pseudo-distance on Γ. We will write $B_\ell(\gamma, r)$ for the ball of center $\gamma \in \Gamma$ and radius r with respect to the pseudo-distance ℓ, and simply $B(\gamma, r)$ when there is no risk of confusion.

Example 1.2. If Γ is generated by some finite subset S, then the *algebraic word length* $L_S : \Gamma \to \mathbf{N}$ is a length function on Γ, where, for $\gamma \in \Gamma$, $L_S(\gamma)$ is the minimal length of γ as a word on the alphabet $S \cup S^{-1}$, that is,

$$L_S(\gamma) = \min\{n \in \mathbf{N} | \gamma = s_1 \dots s_n,\ s_i \in S \cup S^{-1}\}.$$

Let Γ act by isometries on a metric space (X, d). Pick a point $x_0 \in X$ and define $\ell(\gamma) = d(\gamma x_0, x_0)$, this is a length function on Γ. This last example is general in the sense that any length function ℓ comes from a metric on a space X with respect to which Γ acts by isometries. Indeed, if ℓ is a length function on Γ, define the subgroup N of Γ as,

$$N = \{\gamma \in \Gamma | \ell(\gamma) = 0\}$$

and then $X = \Gamma/N$. The map $d : X \times X \to \mathbf{R}_+$, $d(\gamma N, \mu N) = \ell(\mu^{-1}\gamma)$ is a well-defined Γ-invariant metric on X and $\ell(\gamma) = d(\gamma N, N)$ for any $\gamma \in \Gamma$.

Let $H < \Gamma$ be a subgroup of Γ and ℓ a length on Γ. The restriction of ℓ to H induces a length on H that we call *induced length*.

DEFINITION 1.3. Denote by $\mathbf{C}\Gamma$ the set of functions $f : \Gamma \to \mathbf{C}$ with finite support, which is a ring for pointwise addition and convolution:

$$f * g(\gamma) = \sum_{\mu \in \Gamma} f(\mu) g(\mu^{-1}\gamma). \qquad (f, g \in \mathbf{C}\Gamma, \gamma \in \Gamma)$$

We denote by $\mathbf{R}^+\Gamma$ the subset of $\mathbf{C}\Gamma$ consisting of functions with target in $\mathbf{R}^+$. Consider for f in $\mathbf{C}\Gamma$ (or in $\mathbf{R}^+\Gamma$) the following norms:

(a) the usual *ℓ^2 norm*, given by

$$\|f\|_2 = \sqrt{\sum_{\gamma \in \Gamma} |f(\gamma)|^2}$$

which actually comes from a scalar product on $\ell^2\Gamma$, the space of square summable functions on Γ.

(b) the *operator norm*, given by

$$\|f\|_* = \sup\{\|f * g\|_2 \mid \|g\|_2 = 1\}$$

which is the norm of f in $C_r^*\Gamma$, the reduced C*-algebra of Γ, obtained by completing $\mathbf{C}\Gamma$ with respect to the operator norm.

(c) a *weighted ℓ^2 norm*, depending on a parameter $s > 0$ and given by

$$\|f\|_{\ell,s} = \sqrt{\sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s}}.$$

We denote by $H_\ell^s(\Gamma)$ the completion of $\mathbf{C}\Gamma$ with respect to this norm.

Obviously, for $f \in \mathbf{C}\Gamma$ we have that $\|f\|_2 \leq \|f\|_*$, and the following definition is an attempt to give an upper bound to the operator norm.

DEFINITION 1.4 (P. Jolissaint, [**12**]). Let ℓ be a length function on Γ. We say that Γ has *property (RD)* (standing for *Rapid Decay*) with respect to ℓ (or that it satisfies the *Haagerup inequality*), if there exists $C, s > 0$ such that, for each $f \in \mathbf{C}\Gamma$ one has

$$\|f\|_* \leq C \|f\|_{\ell,s}.$$

PROPOSITION 1.5. *Let Γ be a discrete group, endowed with a length function ℓ. Then the following are equivalent:*

1) *The group Γ has property (RD) with respect to ℓ.*
2) *There exists a polynomial P such that, for any $r > 0$ and any $f \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*

$$\|f\|_* \leq P(r) \|f\|_2.$$

3) *There exists a polynomial P such that, for any $r > 0$ and any two functions $f, g \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*

$$\|f * g\|_2 \leq P(r) \|f\|_2 \|g\|_2.$$

4) *There exists a polynomial P such that, for any $r > 0$ and any $f, g, h \in \mathbf{R}_+\Gamma$ so that f vanishes on elements of length greater than r, we have*

$$f * g * h(e) \leq P(r)\|f\|_2\|g\|_2\|h\|_2$$

5) *Any subgroup H in Γ has property (RD) with respect to the induced length.*

PROOF. We start with the equivalence between 1) and 2). Take $f \in \mathbf{C}\Gamma$ with support contained in a ball of radius r, we have:

$$\begin{aligned}\|f\|_* &\leq C\|f\|_{\ell,s} = C\sqrt{\sum_{\gamma\in B(e,r)} |f(\gamma)|^2(\ell(\gamma)+1)^{2s}} \\ &\leq C\sqrt{\sum_{\gamma\in B(e,r)} |f(\gamma)|^2(r+1)^{2s}} = C(r+1)^s\|f\|_2\end{aligned}$$

and thus 2) is satisfied, for the polynomial $P(r) = C(r+1)^s$. Conversely we denote, for $n \in \mathbf{N}$

$$S_n = \{\gamma \in \Gamma | n \leq \ell(\gamma) < n+1\}$$

and compute, for $f \in \mathbf{R}_+\Gamma$:

$$\begin{aligned}\|f\|_* &= \|\sum_{n=0}^{\infty} f|_{S_n}\|_* \leq \sum_{n=0}^{\infty} \|f|_{S_n}\|_* \leq \sum_{n=0}^{\infty} P(n+1)\|f|_{S_n}\|_2 \\ &\leq \sum_{n=0}^{\infty} C(n+1)^k\|f|_{S_n}\|_2 = C\sum_{n=0}^{\infty}(n+1)^{-1}(n+1)^{k+1}\|f|_{S_n}\|_2 \\ &\leq C\sqrt{\sum_{n=0}^{\infty}(n+1)^{-2}}\sqrt{\sum_{n=0}^{\infty}(n+1)^{2k+2}\|f|_{S_n}\|_2^2} \\ &\leq C\frac{\pi}{\sqrt{6}}\sqrt{\sum_{n=0}^{\infty}\sum_{\gamma\in S_n} |f(\gamma)|^2(\ell(\gamma)+1)^{2k+2}} \\ &= C\frac{\pi}{\sqrt{6}}\sqrt{\sum_{\gamma\in\Gamma} |f(\gamma)|^2(\ell(\gamma)+1)^{2k+2}} = C\frac{\pi}{\sqrt{6}}\|f\|_{\ell,k+1}\end{aligned}$$

We finish by noticing that for $f \in \mathbf{C}\Gamma$ if one denotes by $|f|$ the function given by $\gamma \mapsto |f(\gamma)|$ (which is in $\mathbf{R}_+\Gamma$), then $\|f\|_2 = \|\,|f|\,\|_2$ and thus

$$\|f\|_* \leq \|\,|f|\,\|_* \leq P(r)\|\,|f|\,\|_2 = P(r)\|f\|_2.$$

The equivalence between 2) and 3) is rather obvious since for f as before and $g \in \mathbf{R}_+\Gamma$, non zero:

$$\frac{\|f * g\|_2}{\|g\|_2} \leq \|f\|_* \leq P(r)\|f\|_2$$

and thus 2) implies 3). Conversely, take $\epsilon > 0$ and $g_\epsilon \in \mathbf{R}_+\Gamma$ such that $\|f\|_* - \epsilon \leq \frac{\|f*g_\epsilon\|_2}{\|g_\epsilon\|_2}$:

$$\|f\|_* - \epsilon \leq \frac{\|f * g_\epsilon\|_2}{\|g_\epsilon\|_2} \leq P(r)\|f\|_2$$

and since the above inequality holds for any $\epsilon > 0$ we recover point 2).

Let us turn to the equivalence between 3) and 4). To see that 4) implies 3) it is enough to define, for $\gamma \in \Gamma$

$$h(\gamma) = \frac{f * g(\gamma^{-1})}{\|f * g\|_2},$$

and notice that in that case $f * g * h(e) = \|f * g\|_2$ and $\|h\|_2 = 1$. That 3) implies 4) follows from Cauchy-Schwartz inequality:

$$f * g * h(e) = \sum_{\gamma\in\Gamma} f * g(\gamma)\check{h}(\gamma) \leq \|f * g\|_2\|h\|_2$$

where $\check{h}(\gamma) = h(\gamma^{-1})$.

Finally, that 5) implies 1) is trivial since Γ is a subgroup of itself, and the induced length is the original one, and that 2) implies 5) is obvious as well, since if H is a subgroup of Γ, $f \in \mathbf{R}_+H$ supported in a ball of radius r can be viewed in $\mathbf{R}_+\Gamma$, supported in a ball of radius r as well, and

$$\|f\|_{*,H} \leq \|f\|_{*,\Gamma} \leq P(r)\|f\|_{\ell^2\Gamma} = P(r)\|f\|_{\ell^2 H}.$$

□

EXAMPLE 1.6. For a discrete group Γ, the map $\ell_0 : \Gamma \to \mathbf{R}_+$ defined by $\ell_0(\gamma) = 0$ for any $\gamma \in \Gamma$ is a length function, and Γ has property (RD) with respect to ℓ_0 if and only if Γ is finite. Indeed, if Γ has property (RD) with respect to ℓ_0, then there exists a constant C such that for any $f, g \in \mathbf{C}\Gamma$ then $\|f * g\|_2 \leq C\|f\|_2\|g\|_2$, which implies that $\ell^2\Gamma$ is an algebra. This can happen if and only if Γ is finite, see [**23**]. The same statement holds if we just assume ℓ_0 to be bounded.

DEFINITION 1.7. We say that a discrete group Γ has *polynomial growth with respect to a length ℓ* if there exists a polynomial P such that the cardinality of the ball of radius r (denoted by $|B(e, r)|$) is bounded by $P(r)$.

EXAMPLE 1.8 (P. Jolissaint [**12**]). Let Γ be a discrete group endowed with a length function ℓ with respect to which Γ is of polynomial growth, then Γ has property (RD) with respect to ℓ. Indeed, take

$f \in \mathbf{C}\Gamma$ such that $\mathrm{supp}(f) = S_f \subset B(e,r)$, then:

$$\begin{aligned} \|f\|_* &\leq \|f\|_1 = \sum_{\gamma \in \Gamma} |f(\gamma)| = \sum_{\gamma \in S_f} |f(\gamma)| \\ &\leq \sqrt{|S_f|} \sqrt{\sum_{\gamma \in S_f} |f(\gamma)|^2} = \sqrt{|S_f|} \|f\|_2, \end{aligned}$$

the last inequality being just the Cauchy-Schwartz inequality. If Γ is of polynomial growth, then $|S_f| \leq |B(e,r)| \leq P(r)$ and thus $\|f\|_* \leq \sqrt{P(r)}\|f\|_2$.

The following result gives the only known obstruction to property (RD), namely the presence of an amenable subgroup of exponential growth.

Theorem 1.9 (P. Jolissaint [**12**]). *Let Γ be a discrete amenable group. Then Γ has property (RD) with respect to a length function ℓ if and only if Γ is of polynomial growth with respect to ℓ.*

Before giving a proof of this theorem, let us make some remarks concerning amenability of a discrete group Γ. We recall that Γ is *amenable* if the following condition holds (strong Følner condition, see [**21**]): For any $\epsilon > 0$ and any finite set $F \subset \Gamma$, one can find a finite subset V of Γ such that

$$\frac{|FV \Delta V|}{|V|} < \epsilon$$

where $FV \Delta V = \{\gamma \in FV | \gamma \notin V\} \cup \{\gamma \in V | \gamma \notin FV\}$ is the symmetric difference of FV and V.

Let us define, for V and F two finite subsets of Γ, where $e \in F = F^{-1}$:

$$F^{-1}(V) = \{\gamma \in V | \mu^{-1}\gamma \in V \text{ for every } \mu \in F\}$$

so that $V \subset F^{-1}(FV)$ and one gets that:

$$FV = F^{-1}(FV) \cup (FV \Delta V)$$

and thus, for $U = FV$:

$$|F^{-1}(U)| \geq |U| - |FV \Delta V|.$$

We conclude that, in an amenable group Γ, for any finite set F such that $e \in F = F^{-1}$, there exists V a finite subset of Γ such that for $U = FV$ the following holds:

$$\frac{|F^{-1}(U)|}{|U|} \geq 1 - \frac{|FV \Delta V|}{|U|} \geq 1 - \frac{|FV \Delta V|}{|V|} \geq \frac{1}{2} \qquad (\dagger)$$

Proof of Theorem 1.9. We already saw that a group of polynomial growth has property (RD), so let us consider Γ an amenable group of super-polynomial growth, we will define functions contradicting point 3) of Proposition 1.5. For $r \geq 1$, take $F_r \subset B(e,r)$ such that

$e \in F_r = F_r^{-1}$, and U a finite subset of Γ such that (†) holds. We now define the following elements f_r and g of $\mathbf{C}\Gamma$:

$$f_r(\gamma) = \begin{cases} 1 & \text{if } \gamma \in F_r \\ 0 & \text{otherwise} \end{cases}, \quad g(\gamma) = \begin{cases} \frac{1}{\sqrt{|U|}} & \text{if } \gamma \in U \\ 0 & \text{otherwise} \end{cases}$$

so that $\|f_r\|_2 = \sqrt{|F_r|}$, $\|g\|_2 = 1$ and obviously f_r is supported in the ball of radius r. We now compute:

$$\begin{aligned} \|f_r * g\|_2^2 &= \sum_{\gamma\in\Gamma} |\sum_{\mu\in\Gamma} f_r(\mu)g(\mu^{-1}\gamma)|^2 = \sum_{\gamma\in\Gamma} |\sum_{\mu\in F_r} g(\mu^{-1}\gamma)|^2 \\ &\geq \sum_{\gamma\in F_r^{-1}(U)} |\sum_{\mu\in F_r} g(\mu^{-1}\gamma)|^2 = \sum_{\gamma\in F_r^{-1}(U)} |\sum_{\mu\in F_r} \frac{1}{\sqrt{|U|}}|^2 \\ &= \frac{1}{|U|}|F_r^{-1}(U)|\,|F_r|^2 = \frac{|F_r^{-1}(U)|}{|U|}|F_r|\,\|f_r\|_2^2 \\ &\geq \frac{|F_r|}{2}\|f_r\|_2^2. \end{aligned}$$

As soon as Γ is of super-polynomial growth with respect to ℓ, we can choose the sequence of F_r having cardinality growing faster than any polynomial in r, which contradicts 3) and thus Γ cannot have property (RD). In particular this shows that if Γ is amenable and has property (RD) with respect to a length ℓ, then this length has to be proper. □

REMARK 1.10. Denote by $\mathrm{Rad}_L(\Gamma)$ the space of *radial functions*, that is, the functions in $\mathbf{C}\Gamma$ which are constant on elements of equal length. If Γ is of finite type, we can choose characteristic functions on balls of radius r (with respect to the word length) as a sequence of functions contradicting property (RD) in the above proof, and those are in particular radial. In Proposition 6 Section 3 of [**29**], A. Valette already showed that an amenable group Γ has to be of polynomial growth if in $\mathrm{Rad}_L(\Gamma)$ the Haagerup inequality is satisfied. In the same article, it is shown that this inequality holds for $\mathrm{Rad}_L(\Gamma)$, where Γ is any $\tilde{A}_n$ group, those including uniform lattices in $SL_n(\mathbf{Q}_p)$. However, $\mathrm{Rad}_L(\Gamma)$ is a too small subspace in $\mathbf{C}\Gamma$ to deduce property (RD) for Γ itself.

A very short proof of Theorem 1.9 is given in [**30**] using the weak containment of the trivial representation in the regular representation as a definition for amenability, and applying it to the above given functions f_r. This proof is somehow similar to P. Jolissaint's original proof. I decide to keep the above given proof of Theorem 1.9 because Følner condition allows to give explicitly the functions on which the f_r's reach their operator norm.

The above stated result, combined with P. Jolissaint's Theorem 1.9 and with the following unpublished result

PROPOSITION 1.11 (Unpublished, E. Leuzinger and C. Pittet). *Any non-uniform lattice in a higher rank simple Lie group contains a solvable subgroup with exponential growth (with respect to its word length).*

led to the statement of Conjecture 1. Let us remark that according to A. Lubotzky, S. Mozes and M. S. Raghunathan in [**17**], any non uniform lattice Γ in a higher rank simple Lie group contains a cyclic subgroup which is of exponential growth with respect to the generators of Γ, and this fact, combined with Proposition 1.5 part 5) and Theorem 1.9 already creates an obstruction to property (RD) for non-uniform lattices in higher rank simple Lie groups. In particular, it means that $SL_n(\mathbf{Z})$ doesn't have property (RD) as soon as $n \geq 3$.

We will now proceed with further results concerning property (RD) and length functions on groups.

DEFINITION 1.12. Let ℓ_1 and ℓ_2 be two length functions on a discrete group Γ. We say that ℓ_1 *dominates* ℓ_2 (and write $\ell_1 \geq \ell_2$) if there exists two integers C and k such that, for any $\gamma \in \Gamma$ one has:

$$\ell_2(\gamma) \leq C(1 + \ell_1(\gamma))^k.$$

We will say that ℓ_1 is *equivalent* to ℓ_2 if furthermore ℓ_2 dominates ℓ_1.

REMARK 1.13. If Γ is a finitely generated discrete group, then the algebraic word length L_S associated to a finite generating set S of Γ dominates any other length ℓ on Γ. Indeed, for $\gamma \in \Gamma$, let $s_1 \dots s_n$ be a minimal word in the letters of S, then:

$$\ell(\gamma) = \ell(s_1 \dots s_n) \leq \sum_{i=1}^{n} \ell(s_i) \leq An = AL_S(\gamma)$$

where $A = \max_{s \in S}\{\ell(s)\}$ is finite since S is finite.

If a length ℓ_1 dominates another length ℓ_2 on a discrete group Γ and if Γ has property (RD) with respect to ℓ_2, then Γ has property (RD) with respect to ℓ_1 as well. Indeed, denote for $i = 1, 2$ and $r \in \mathbf{R}_+$ by $B_i(e, r)$ the ball of radius r for the length ℓ_i, centered at e and take $f \in \mathbf{C}\Gamma$ such that $S_f \subset B_1(e, r)$. This means that for $\gamma \in \Gamma$ such that $f(\gamma) \neq 0$, then $\ell_1(\gamma) \leq r$, and thus

$$\ell_2(\gamma) \leq C(1 + \ell_1(\gamma))^k \leq C(1 + r)^k,$$

which implies that $S_f \subset B_2(e, C(1 + r)^k)$ and applying point 2) of Proposition 1.5 we get that

$$\|f\|_* \leq P(C(1 + r)^k)\|f\|_2.$$

Thus Γ has property (RD) with respect to the length ℓ_1, and the polynomial is given by $Q(r) = P(C(1 + r)^k)$.

In particular, this shows that changing the set of generators on a finitely generated group will not change the degree of the polynomial involved, provided the considered generating sets are finite, and thus we can talk about word length without really caring of the chosen generating set.

This remark says in particular that a finitely generated group Γ has property (RD) with respect to the word length as soon as it has property (RD) for any other length. We will see later that it is mainly property (RD) with respect to the word length which is useful, but other lengths on groups are very important as well, as we shall see in the following proposition. In P. Jolissaint's Proposition 2.1.9 in [**12**], he gives sufficient conditions to ensure property (RD) for an extension of groups having property (RD) with respect to the word length. We give a necessary and sufficient condition to property (RD) in the particular case of split extensions:

PROPOSITION 1.14. *Let Γ be a discrete finitely generated group having property (RD) with respect to the word length, and let E be a split extension of a discrete group G by Γ:*

$$\{e\} \longrightarrow G \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } \Gamma \longrightarrow \{e\}$$

If E is finitely generated, then E has property (RD) with respect to the word length if, and only if $i(G)$ has property (RD) with respect to a length dominated by the induced word length of E.

Before proceeding with the proof, we recall some basic facts about split extensions (or semi-direct products) that can be found in [**25**]. Such a group E can be described as follows: as a set E is just $G \times \Gamma$, and $i : G \to E$ is given by $i(a) = (a, e)$ (for any $a \in G$), whereas $\pi : E \to \Gamma$ by $\pi(a, \gamma) = \gamma$ (for any $(a, \gamma) \in E$), and the group law is given by:

$$(a, \gamma) \cdot (b, \mu) = (a\varphi_\gamma(b), \gamma\mu)$$

where $\varphi : \Gamma \to \mathrm{Aut}(G)$ is a group homomorphism. It is a simple computation to see that this determines a group law on E, and that the inverse of an element $(a, \gamma) \in E$ is given by $(\varphi_{\gamma^{-1}}(a)^{-1}, \gamma^{-1})$.

PROOF OF PROPOSITION 1.14. That $i(G)$ has property (RD) with respect to the induced word length follows from Proposition 1.5, so let us do the other implication. We assume that G has property (RD) with respect to the induced length of E, and choose $f, g \in \mathbf{C}E$ such that f is supported in a ball of radius r. We now compute, for $(a, \gamma) \in E$:

$$\begin{aligned} f * g(a, \gamma) &= \sum_{(b,\mu)\in E} f(b, \mu) g\left((\varphi_{\mu^{-1}}(b)^{-1}, \mu^{-1}) \cdot (a, \gamma)\right) \\ &= \sum_{\mu\in\Gamma} \left(\sum_{b\in G} f_\mu(b) g_{\mu^{-1}\gamma} \circ \varphi_{\mu^{-1}}(b^{-1}a) \right) = \sum_{\mu\in\Gamma} \left(f_\mu * g'_{(\mu,\gamma)}\right)(a) \end{aligned}$$

where we defined $f_\mu(a) = f(a,\mu)$ and $g'_{(\mu,\gamma)} = g_{\mu^{-1}\gamma} \circ \varphi_{\mu^{-1}}$. We proceed with the computation:

$$\begin{aligned} \|f * g\|^2_{\ell^2 E} &= \sum_{(a,\gamma)\in E} \left| \sum_{\mu\in\Gamma} \left(f_\mu * g'_{(\mu,\gamma)}\right)(a) \right|^2 = \sum_{\gamma\in\Gamma} \left(\| \sum_{\mu\in\Gamma} f_\mu * g'_{(\mu,\gamma)} \|_{\ell^2 G} \right)^2 \\ &\le \sum_{\gamma\in\Gamma} \left(\sum_{\mu\in\Gamma} \|f_\mu * g'_{(\mu,\gamma)}\|_{\ell^2 G} \right)^2 \end{aligned}$$

Let us look at the support of f_μ: Since E is finitely generated, we can find a finite number of elements $\{(a_i, s_i)\}$ $(i = 1, \dots, n)$ which generate E, where $a_i \in G$, $s_i \in \Gamma$. We choose $T = \bigcup_{i=1}^n (a_i, s_i) \bigcup_{i=1}^n (e, s_i)$ as a generating set for E. Take $x \in \operatorname{supp}(f_\mu)$, then (x,μ) is in the support of f, that is, $\ell_E(x,\mu) \le r$ and thus in particular $\ell_E(e,\mu) \le r$ (indeed, $(x,\mu) = (a_1, s_1)\dots(a_r, s_r)$ implies that $(e,\mu) = (e, s_1)\dots(e, s_r)$).

Now, by assumption, there is a length ℓ on G with respect to which G has property (RD) and such that, for any $x \in G$ the following holds:

$$\ell(x) \le C(1 + \ell_E(x,e))^k$$

for some fixed constants $C, k \ge 0$, and thus:

$$\begin{aligned} \ell(x) &\le C\left(1 + \ell_E\left((x,e)\cdot(e,\mu)\cdot(e,\mu)^{-1}\right)\right)^k \\ &= C\left(1 + \ell_E\left((x,\mu)\cdot(e,\mu)^{-1}\right)\right)^k \\ &\le C(1 + \ell_E(x,\mu))^k (1 + \ell_E(e,\mu))^k \le C(1+r)^{2k}, \end{aligned}$$

so that the support of f_μ is contained in a ball of radius $C(1+r)^{2k}$ and thus:

$$\|f * g\|^2_{\ell^2 E} \le \sum_{\gamma\in\Gamma} \left(\sum_{\mu\in\Gamma} P_G(C(1+r)^{2k}) \|f_\mu\|_{\ell^2 G} \|g'_{(\mu,\gamma)}\|_{\ell^2 G} \right)^2 .$$

Finally, define $\tilde f, \tilde g \in \mathbf{C}\Gamma$ by $\tilde f(\mu) = \|f_\mu\|_{\ell^2 G}$ and $\tilde g(\mu) = \|g_\mu\|_{\ell^2 G}$, so that clearly $\|\tilde f\|_{\ell^2\Gamma} = \|f\|_{\ell^2 E}$ and $\|\tilde g\|_{\ell^2\Gamma} = \|g\|_{\ell^2 E}$. Notice that $\tilde f$ is supported on a ball of radius r. Indeed, if μ is in the support of $\tilde f$, then $\|f_\mu\|_2$ is non zero, which means that there exists an $x \in G$ such that (x,μ) is in the support of f, which is contained in a ball of radius r in E, and $\ell_\Gamma(\mu) \le \ell_E(e,\mu) \le \ell_E(x,\mu)$. Concerning g, we see that:

$$\begin{aligned} \|g'_{(\mu,\gamma)}\|_{\ell^2 G} &= \sqrt{\sum_{a\in G} |g_{\mu^{-1}\gamma} \circ \varphi_{\mu^{-1}}(a)|^2} \\ &= \sqrt{\sum_{a\in G} |g_{\mu^{-1}\gamma}(a)|^2} = \|g_{\mu^{-1}\gamma}\|_{\ell^2 G} = \tilde g(\mu^{-1}\gamma) \end{aligned}$$

(we performed the change of variable $a \mapsto \varphi_{\mu^{-1}}(a)$). Going back to the computation of $\|f * g\|^2_{\ell^2 E}$ we now get that:

$$\begin{aligned}
\|f * g\|^2_{\ell^2 E} &\leq P_G(C(1+r)^{2k})^2 \sum_{\gamma\in\Gamma}\left(\sum_{\mu\in\Gamma}\tilde{f}(\mu)\tilde{g}(\mu^{-1}\gamma)\right)^2 \\
&= P_G(C(1+r)^{2k})^2\|\tilde{f} * \tilde{g}\|^2_{\ell^2\Gamma} \\
&\leq P_G(C(1+r)^{2k})^2 P_\Gamma(r)^2\|\tilde{f}\|^2_{\ell^2\Gamma}\|\tilde{g}\|^2_{\ell^2\Gamma}
\end{aligned}$$

So we conclude setting $P_E(r) = P_G(C(1+r)^{2k})P_\Gamma(r)$. □

Remark 1.15. Under the assumptions of the above proposition, if furthermore G is finitely generated and has property (RD) with respect to the word length, and if $\varphi : \Gamma \to \mathrm{Aut}(G)$ has polynomial amplitude as defined by P. Jolissaint in [**12**] (meaning that there exists two constants C and k such that $\ell_G(\varphi_\gamma(s)) \leq C(\ell_\Gamma(\gamma)+1)^k$ for every $\gamma \in \Gamma$ and every s in a generating set for G), then the induced length on G is equivalent to the word length, and thus E has property (RD), as already shown by P. Jolissaint.

Example 1.16. Let $\mathbf{F}_2$ denote the free group on two generators a and b, and consider α the automorphism of $\mathbf{F}_2$ given by $a \mapsto a^2b$, and $b \mapsto ab$. Let us look at $\Gamma = \mathbf{F}_2 \rtimes_\alpha \mathbf{Z}$. The action of $\mathbf{Z}$ on $\mathbf{F}_2$ has not polynomial amplitude, but the automorphism α is hyperbolic (meaning that there is no conjugacy class of $\mathbf{F}_2$ stabilized by α) and thus Γ is hyperbolic due to a result by P. Brinkmann (see [**1**]). Hence Γ has property (RD).

We now discuss the case of central extensions. We recall (see [**25**]) that for a central extension of an abelian group $(G,+)$ by a group Γ

$$\{e\} \longrightarrow G \xrightarrow{i} E \xrightarrow{\pi} \Gamma \longrightarrow \{e\}$$

then E is again just $G \times \Gamma$ as a set, and the group law is given by

$$(a,\gamma)\cdot(b,\mu) = (a+b+\beta(\gamma,\mu),\gamma\mu)$$

where $\beta : \Gamma\times\Gamma \to G$ is a map such that

- $\beta(\gamma,e) = \beta(e,\gamma) = e$
- for any $\gamma,\mu,\delta \in \Gamma$, the cocycle condition is satisfied:

$$\beta(\gamma,\mu)+\beta(\gamma\mu,\delta) = \beta(\mu,\delta)+\beta(\gamma,\mu\delta).$$

It is a simple computation to see that this determines a group law on E, and that the inverse of an element $(a,\gamma) \in E$ is given by $(-a-\beta(\gamma,\gamma^{-1}),\gamma^{-1})$.

Definition 1.17. We say that a cocycle $\beta : \Gamma\times\Gamma \to G$ is of *polynomial growth* if there exists two constants k and C such that for any $\gamma,\mu \in \Gamma$ of length shorter than r, then $\ell_G(\beta(\gamma,\mu)) \leq C(1+r)^k$

The following proposition is a particular case of P. Jolissaint's Proposition 2.1.9 in [**12**].

Proposition 1.18 (P. Jolissaint). *let G and Γ be two finitely generated groups, G abelian and Γ with property (RD) with respect to the word length. If $\beta : \Gamma \times \Gamma \to G$ is a cocycle of polynomial growth, then the central extension E of G by Γ has property (RD) with respect to the word length.*

Proof. First notice that a finitely generated abelian group is automatically of polynomial growth for the word length, and thus has property (RD) with respect to this length.

Let $T = T^{-1}$ be a generating set for Γ and $S = S^{-1}$ be a generating set for G. Then $U = \{(s, e)|s \in S\} \cup \{(e, t)|t \in T\}$ is a finite generating set for E. Let us first show that for any $a \in G$ then $L_S(a) \leq C'(1 + L_U(a, e))^{k'}$ (for some fixed constants C' and k'). To do this, assume that (a, e) is of length r in E and write $(a, e) = (a_1, \gamma_1) \dots (a_r, \gamma_r)$ where (a_i, γ_i) belong to the generating set U, for $i = 1, \dots, r$. Then, defining $\lambda_i = \gamma_1 \dots \gamma_i$ for $i = 1, \dots, r$ we get:

$$(a, e) = (a_1, \gamma_1) \dots (a_r, \gamma_r) = (\sum_{i=1}^{r} a_i + \sum_{i=1}^{r-1} \beta(\lambda_i, \gamma_{i+1}), e)$$

and $a = \sum_{i=1}^{r} a_i + \sum_{i=1}^{r-1} \beta(\lambda_i, \gamma_{i+1})$, so that

$$L_S(a) \leq \sum_{i=1}^{r} L_S(a_i) + \sum_{i=1}^{r-1} L_S(\beta(\lambda_i, \gamma_{i+1})) \leq r + (r-1)C(1+r)^k \leq C'(1+r)^{k'}$$

choosing $C' = C + 1$ and $k' = k + 1$. We proceed with computations which are very similar to those in the proof of the previous proposition. For $f, g \in \mathbf{C}E$,

$$\|f * g\|^2_{\ell^2 E} \leq \sum_{\gamma \in \Gamma} \left(\sum_{\mu \in \Gamma} \|f_\mu * g'_{(\mu,\gamma)}\|_{\ell^2 G} \right)$$

where we defined $f_\mu(a) = f(a, \mu)$ and $g'_{(\mu,\gamma)}(a) = g_{\mu^{-1}\gamma}(a - \beta(\mu, \mu^{-1}))$. Now assume that the support of f is contained in a ball of radius r and for $\mu \in \Gamma$, let us look at the support of f_μ. Take a in the support of f_μ, then $L_U(a, \mu) \leq r$ and thus

$$L_S(a) \leq C'(1 + r)^{k'}$$

because of the computation in the beginning of this proof, so that

$$\|f * g\|^2_{\ell^2 E} \leq \sum_{\gamma \in \Gamma} \left(\sum_{\mu \in \Gamma} P_G(C'(1 + r)^{k'}) \|f_\mu\|_{\ell^2 G} \|g'_{(\mu,\gamma)}\|_{\ell^2 G} \right)^2 .$$

Finally, define $\tilde{f}, \tilde{g} \in \mathbf{C}\Gamma$ as in the proof of Proposition 1.14, and notice that $\tilde{f}$ is supported on a ball of radius r. Indeed, if μ is in the support of $\tilde{f}$, then there exists an $x \in G$ such that (x, μ) is in the support of f, which is contained in a ball of radius r in E. Writing $(x, \mu) = (a_1, \gamma_1) \dots (a_r, \gamma_r)$ we see in particular that $\mu = \gamma_1 \dots \gamma_r$, i.e. the length of μ in Γ (with respect to the generating set T) is shorter than r. Concerning g, we have that:

$$\|g'_{(\mu,\gamma)}\|_{\ell^2 G} = \|g_{\mu^{-1}\gamma}\|_{\ell^2 G} = \tilde{g}(\mu^{-1}\gamma)$$

(we performed the change of variable $a \mapsto a - \beta(\mu, \mu^{-1})$). Going back to the computation of $\|f * g\|^2_{\ell^2 E}$ we now get that:

$$\begin{aligned} \|f * g\|^2_{\ell^2 E} &\leq P_G(C'(1+r)^{k'})^2 \|\tilde{f} * \tilde{g}\|^2_{\ell^2 \Gamma} \\ &\leq P_G(C'(1+r)^{k'})^2 P_\Gamma(r)^2 \|f\|^2_{\ell^2 E} \|g\|^2_{\ell^2 E}. \end{aligned}$$

We conclude setting $P_E(r) = P_G(C'(1+r)^{k'})P_\Gamma(r)$. □

The following corollary was previously obtained by G. A. Noskov in [**20**].

COROLLARY 1.19. *Any central extension of a finitely generated abelian group G by a finitely generated Gromov hyperbolic group has property (RD) with respect to the word length.*

PROOF. A finitely generated Gromov hyperbolic group Γ has property (RD) by [**8**]. Since by [**18**] any 2-cocycle on such Γ is (up to a coboundary) equivalent to a bounded cocycle, any central extension is isomorphic to a central extension described by a bounded cocycle, which is in particular trivially of polynomial growth. □

As a logical sequel of Proposition 1.14 we mention the following result, also already proven in [**12**]

PROPOSITION 1.20. *Let G and Γ be two discrete groups of finite type, and*

$$\{e\} \longrightarrow G \xrightarrow{i} E \xrightarrow{\pi} \Gamma \longrightarrow \{e\}$$

a split extension of G by Γ. If G (respectively Γ) is finite, then E has property (RD) with respect to the word length if, and only if Γ (respectively G) has property (RD) with respect to the word length.

PROOF. To start with, assume G finite. If Γ has property (RD) with respect to the word length, then E has it as well by Proposition 1.14 since finite groups have property (RD) for any length. Conversely, take $f \in \mathbf{C}\Gamma$ and define $\overline{f} \in \mathbf{C}E$ by $\overline{f} = f \circ \pi$, so that $\|\overline{f}\|_{\ell^2 E} = \sharp G \|f\|_{\ell^2 \Gamma}$ and:

$$\begin{aligned} (\overline{f} * \overline{g})(x) &= \sum_{y \in E} f \circ \pi(y) g \circ \pi(y^{-1}x) = \sum_{\gamma \in \Gamma} \sum_{a \in G} f(\pi(a,\gamma)) g(\pi(a,\gamma)^{-1}\pi(x)) \\ &= \sharp G \sum_{\gamma \in \Gamma} f(\gamma) g(\gamma^{-1}\pi(x)) = \sharp G (f * g)(\pi(x)) = \sharp G (\overline{f * g})(x) \end{aligned}$$

and thus

$$\|\overline{f*g}\|_{\ell^2 E} = \sqrt{\sum_{x\in E} |(\overline{f*g})(x)|^2} = \sqrt{\sum_{x\in E} |\frac{1}{\sharp G}(\overline{f}*\overline{g})(x)|^2} = \frac{1}{\sharp G}\|\overline{f}*\overline{g}\|_{\ell^2 E}$$

If a function f in $\mathbf{C}\Gamma$ is supported in a ball of radius r in Γ then there exists a constant C such that $\overline{f}$ is supported in a ball of radius $r+C$ in E which means that finally we conclude that, for any $g \in \mathbf{C}\Gamma$:

$$\begin{aligned}\|f*g\|_{\ell^2\Gamma} &= \frac{1}{\sharp G}\|\overline{f*g}\|_{\ell^2 E} = \frac{1}{(\sharp G)^2}\|\overline{f}*\overline{g}\|_{\ell^2 E} \\ &\leq \frac{1}{(\sharp G)^2}P(r+C)\|\overline{f}\|_{\ell^2 E}\|\overline{g}\|_{\ell^2 E} = P(r+C)\|f\|_{\ell^2\Gamma}\|g\|_{\ell^2\Gamma}\end{aligned}$$

Now assume that Γ is finite. Because of Proposition 1.14 it is enough to show that the word length on G is equivalent to the induced length of E, and this is done as follows: let $S = S^{-1}$ be a generating set for G and set

$$C = \max_{\gamma,\mu\in\Gamma, a\in S}\{\ell_G(\alpha_{(\gamma,\mu)}(a))\}$$

which is a finite number since S and Γ are finite (where α is as defined for the proof of 1.14). Take $a \in G$ with $\ell_E(a,e) = r$ so that

$$(a,e) = (a_1,\gamma_1)\dots(a_r,\gamma_r) = (\prod_{i=0}^{r-1}\alpha_{(\lambda_i,\gamma_{i+1})}(a_{i+1}), e)$$

(where $\lambda_i = \gamma_1\dots\gamma_i$ for $i = 1,\dots,r$, $\lambda_0 = e$) and thus

$$\ell_G(a) = \ell_G\left(\prod_{i=0}^{r-1}\alpha_{(\lambda_i,\gamma_{i+1})}(a_{i+1})\right) \leq \sum_{i=0}^{r-1}\ell_G\left(\alpha_{(\lambda_i,\gamma_{i+1})}(a_{i+1})\right) \leq C\ell_E(a,e).$$

□

What is property (RD) good for?

In this section we will be discussing two contexts in which it can be useful to know whether a discrete group Γ has property (RD), the first one will be the Baum-Connes conjecture and the second one will be the theory of random walks. We will not make any attempt to describe what the Baum-Connes conjecture really is about but we just state it. To do so, we will denote (for a discrete group Γ) by $\underline{E}\Gamma$ the classifying space for proper actions and by $RK_*^\Gamma(\underline{E}\Gamma)$ its equivariant K-homology (for $* = 0,1$). P. Baum and A. Connes defined a map μ_* from $RK_*^\Gamma(\underline{E}\Gamma)$ to $K_*(C_r^*\Gamma)$, where $K_*(C_r^*\Gamma)$ is the K-theory of the C*-algebra $C_r^*\Gamma$, which is called *the assembly map* and formulated the following:

CONJECTURE 2 (the Baum-Connes Conjecture). *Let* Γ *be a discrete group. The assembly map*

$$\mu_*^\Gamma : RK_*^\Gamma(\underline{E}\Gamma) \to K_*(C_r^*\Gamma)$$

is an isomorphism.

We refer to [**30**] for a reader friendly description of this conjecture, a wide bibliography concerning it and an overview of some related conjectures. We now will briefly describe where, in the work of V. Lafforgue in [**16**] on Conjecture 2, property (RD) comes in.

DEFINITION 1.21. Let ℓ be a length function on Γ, define

$$H_\ell^\infty(\Gamma) = \bigcap_{s\geq 0} H_\ell^s(\Gamma) = \bigcap_{s\geq 0}\{f : \Gamma \to \mathbf{C} | \|f\|_{\ell,s} < \infty\}.$$

These are the functions of *rapid decay* on Γ: they decay faster than the inverse of any polynomial in L.

We can now state some useful properties of the reduced C*-algebras of groups having property (RD).

THEOREM 1.22 (P. Jolissaint, [**12**]). *If a group* Γ *has property (RD) with respect to some length function* ℓ*, then*

1. *The algebra* $H_\ell^\infty(\Gamma)$ *is a dense subalgebra of* $C_r^*\Gamma$*.*
2. *The inclusion* $H_\ell^\infty(\Gamma) \hookrightarrow C_r^*\Gamma$ *induces isomorphisms in* K*-theory.*

PROPOSITION 1.23 (V.Lafforgue [**14**]). *Assume that* Γ *has property (RD) with respect to* ℓ*. Then* $H_\ell^s(\Gamma)$ *is a Banach algebra for* s *large enough.*

COROLLARY 1.24. *If* Γ *has property (RD) with respect to* ℓ*, then, for* s *large enough, the inclusion* $H_\ell^s(\Gamma) \hookrightarrow C_r^*\Gamma$ *induces epimorphisms in* K*-theory.*

DEFINITION 1.25 (V.Lafforgue). A Banach algebra $\mathcal{A}\Gamma$ is an *unconditional completion* of $\mathbf{C}\Gamma$ if it contains $\mathbf{C}\Gamma$ as a dense subalgebra and if, for $f_1, f_2 \in \mathbf{C}\Gamma$ such that $|f_1(\gamma)| \leq |f_2(\gamma)|$ for all $\gamma \in \Gamma$, we have

$$\|f_1\|_{\mathcal{A}\Gamma} \leq \|f_2\|_{\mathcal{A}\Gamma}.$$

Taking $f_1 = |f_2|$, we notice that $\|f\|_{\mathcal{A}\Gamma} = \||f|\|_{\mathcal{A}\Gamma}$ for all $f \in \mathbf{C}\Gamma$.

REMARK 1.26. The Banach algebra $\ell^1\Gamma$ is an unconditional completion, and if Γ has property (RD) with respect to a length function ℓ, then for s large enough, $H_\ell^s(\Gamma)$ is a convolution algebra and an unconditional completion. The reduced C*-algebra is in general not an unconditional completion, even for $\Gamma = \mathbf{Z}$.

V. Lafforgue constructs a map $\mu_{\mathcal{A}}$ from $RK_*^\Gamma(\underline{E}\Gamma)$ to the K-theory $K_*(\mathcal{A}\Gamma)$ of the unconditional completion $\mathcal{A}\Gamma$, which is compatible with the assembly map μ_*. He then defines a class $\mathcal{C}'$ of groups, closed by products, containing (among many others) discrete groups acting properly and isometrically either on a simply connected Riemannian manifold with non positive curvature bounded from below, or on an affine uniformly locally finite Bruhat-Tits building, and proves the following

THEOREM 1.27 (V.Lafforgue [**16**]). *For any group belonging to the class $\mathcal{C}'$ and for any unconditional completion $\mathcal{A}\Gamma$ of $\mathbf{C}\Gamma$ the map $\mu_{\mathcal{A}}$ is an isomorphism.*

In view of the properties of $H^s_\ell(\Gamma)$ for s large enough, this theorem gives

THEOREM 1.28 (V.Lafforgue [**16**]). *The Baum-Connes conjecture holds for any property (RD) group belonging to the class $\mathcal{C}'$.*

Due to these results, the importance of Conjecture 1 in the context of the Baum-Connes Conjecture is now clear. The class of groups $\mathcal{C}'$ contains also discrete groups acting properly and isometrically on a weakly geodesic and strongly bolic metric space (see [**16**] for the definition of strong bolicity) and recently I. Mineyev and G. Yu in [**19**] established the following

THEOREM 1.29 (I. Mineyev and G. Yu). *Every discrete hyperbolic group Γ admits a Γ-invariant metric d which is quasi-isometric to the word metric and such that the metric space (Γ, d) is weakly geodesic and strongly bolic.*

which is the main step in their proof of the Baum-Connes conjecture for hyperbolic groups and their subgroups. This also shows that property (RD) can be important for groups which are not covered by Conjecture 1.

Let us now briefly discuss property (RD) in the context of random walks. Again we will not even make an attempt to describe what a random walk is but we will just state a consequence of property (RD) that has been used in this context.

LEMMA 1.30. *Let Γ be a discrete group having property (RD) with respect to a length function ℓ and take $\{f_n\}_{n\geq 1}$ a sequence of elements in $\mathbf{C}\Gamma$. Assume that for each $n \geq 1$ the support of f_n is contained in the ball of radius n in Γ. Then*

$$\limsup_{n\to\infty} \|f_n\|_*^{1/n} = \limsup_{n\to\infty} \|f_n\|_2^{1/n}.$$

PROOF. Since f_n is supported on the ball of radius n in Γ we have that

$$\|f_n\|_2 \leq \|f_n\|_* \leq P(n)\|f_n\|_2$$

so that for any $n \in \mathbf{N}$

$$\left(\|f_n\|_2\right)^{1/n} \leq \left(\|f_n\|_*\right)^{1/n} \leq P(n)^{1/n}\left(\|f_n\|_2\right)^{1/n}$$

and thus we conclude since $\lim_{n\to\infty} P(n)^{1/n} = 1$. □

This lemma has been used by R. Grigorchuck and T. Nagnibeda in [**6**] to compute the convergence radius of the complete growth serie of a hyperbolic group.

Another application is as follows. Take Γ a finitely generated group and S a finite generating set. Denote by h the characteristic function of S, that is, $h(s) = 1$ for any $s \in S$ and 0 otherwise. Denote by

$$h^{\underline{n}} = \underbrace{h * \cdots * h}_{n \text{ terms}}$$

for $n \in \mathbf{N}$, then the map

$$\begin{array}{rcl} \Gamma \times \Gamma & \to & \mathbf{R} \\ (\gamma, \mu) & \mapsto & h^{\underline{n}}(\gamma^{-1}\mu) \end{array}$$

computes the number of paths of length n connecting two elements γ and μ. One question in this context is to compute the spectral radius $r(h)$ of the function h, which is given by

$$r(h) = \lim_{n\to\infty} \left(\|h^{\underline{n}}\|_*\right)^{1/n}.$$

In case where the generating system S is symmetric, then $r(h) = \lim_{n\to\infty}\left(\|h^{\underline{n}}\|_2\right)^{1/n}$, but if S is not symmetric it is not necessarily so and the previous lemma says that if the group Γ has property (RD), since $h^{\underline{n}}$ is supported on the ball of radius n in Γ we have indeed that $r(h) = \lim_{n\to\infty}\left(\|h^{\underline{n}}\|_2\right)^{1/n}$. This has been used by P. de la Harpe, G. Robertson and A. Valette in [**9**] to establish that for Γ a group with property (RD), then $r(h) = |S|^{-1/2}$ if and only if S generates a free semi-group.

CHAPTER 2

Uniform lattices in products of rank one Lie groups

In the first section we will state two properties for a metric space and show that any group acting simply transitively on a discrete metric space having those two properties has property (RD). The first property, called (H_δ), is due to V. Lafforgue and gives a polynomial bound on the number on δ-paths between any two points. We will mostly be using a stronger property that we will call $(\underline{H})$. The second property, called (L) would read: "there is a $\delta \geq 0$ such that every triple of points is δ-thin" in V. Lafforgue terminology, where a triple x, y, z is said *δ-thin* if there exists a point t such that the paths xty, ytz and ztx are δ-paths. We chose to change the terminology from *δ-thin* to *δ-retractable* because we will see that in particular $\mathbf{R}^2$ endowed with the ℓ^1 norm will have property (L). In the second section of this chapter we will prove that any uniform net in a finite product of hyperbolic spaces has both properties $(\underline{H})$ and (L), and deduce that any cocompact lattice in the isometry group of a finite product of hyperbolic spaces has property (RD). The third section of this chapter is due to a remark made by N. Higson which shows that Coxeter groups have property (RD).

Recall that a metric space (X, d) is said to be *geodesic* if for any two points $x, y \in X$ there exists at least one geodesic path going from x to y. In this work we will always assume our metric spaces to be geodesics. A *geodesic triangle* Δ is the data of three points $x, y, x \in X$ together with three (possibly non unique) geodesics γ_{xy}, γ_{yz} and γ_{zx} joining them.We call *inner diameter* of Δ the minimal diameter of the sets $\{u, v, w\}$, with $u \in \gamma_{xy}, v \in \gamma_{yz}, w \in \gamma_{zx}$. Let $\delta \geq 0$, the geodesic metric space X is said *Gromov hyperbolic* (or *δ-hyperbolic*) if any geodesic triangle in X has *inner diameter* bounded by δ. This is one of the many equivalent definitions of hyperbolicity, see [**5**].

Two properties for a metric space

Definition 2.1. Let (X, d) be a metric space and $\delta \geq 0$. For any finite sequence of points $x_1, \ldots, x_n \in X$, we say that $x_1 \ldots x_n$ is a *δ-path* if

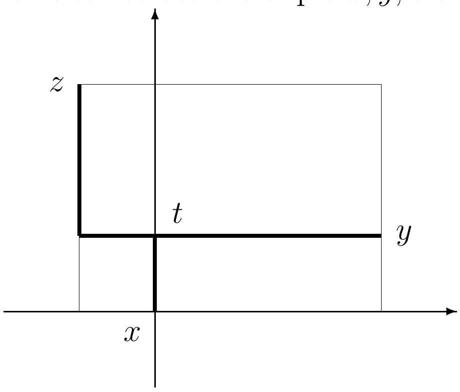
$$d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leq d(x_1, x_n) + \delta$$

and that three points $x, y, z \in X$ form *δ-retractable triple* if there exists $t \in X$ such that the paths xty, ytz and ztx are δ-paths. We will say that X satisfies *property* (L_δ) if there exists a $\delta \geq 0$ such that any triple is δ-retractable. Notice that if a triple is δ-retractable, then it is δ'-retractable for any $\delta' \geq \delta$, and we will be talking of property (L) when the δ doesn't matter.

EXAMPLE 2.2. Let us briefly explain why $(\mathbf{R}^2, \ell^1)$ has property (L_0). Let x, y, z be three points in $\mathbf{R}^2$. With no loss of generality, we can assume that $x = (0,0)$ and that $y = (y_1, y_2)$ with $y_1, y_2 \geq 0$. We write $z = (z_1, z_2)$, then $t = (t_1, t_2)$ with

$$t_i = \begin{cases} \min\{y_i, z_i\} & \text{if} \quad z_i \geq 0 \\ 0 & \text{if} \quad z_i \leq 0. \end{cases}$$

for $i = 1, 2$ is the sought point to retract the triple x, y, z on. Here is a picture of the situation:



DEFINITION 2.3. Let $\delta \geq 0$; a discrete metric space (X, d) satisfies *property* (H_δ) if there exists a polynomial P_δ (depending on δ) such that for any $r \in \mathbf{R}_+$, $x, y \in X$ one has

$$\sharp\{t \in X \text{ such that } d(x,t) \leq r, \text{ and } xty \ \delta - \text{path}\} \leq P_\delta(r).$$

We say that (X, d) satisfies *property* $(\underline{H})$ if it satisfies (H_δ) for every $\delta \geq 0$. Notice that if X has property (H_δ), then it will have property (H_η) for any $\delta \geq \eta$, whereas if X has property (L_δ), then it will have property (L_η) for any $\delta \leq \eta$.

REMARK 2.4. V. Lafforgue (in [**14**], Lemma 3.4) proved that any cocompact lattice in a semi-simple Lie group G has property $(\underline{H})$, provided that the distance satisfies some conditions that we will now describe. Let us start with some sotations; let K be a maximal compact subgroup in G, $A = \exp(\mathfrak{a})$ with $\mathfrak{a}$ a maximal abelian subspace in the eigenspace of the eigenvalue one for the Cartan involution on the Lie algebra of G, $A^+ = \exp(\mathfrak{a}^+)$, where $\mathfrak{a}^+ \subset \mathfrak{a}$ is the set of all elements on which a chosen set of positive restricted roots is positive (see [**13**]). Now assume that d is a distance on G/K is given as follows:

$$d(x, y) = L(\log a)$$

where $x^{-1}y = KaK$ with $a \in \overline{A^+}$ and L is such that

$$L = \sum_{i=1}^{k} n_i \alpha_i$$

with all the n_i's non zero and $\{\alpha_1, \dots, \alpha_n\}$ is a basis for the restricted root system Σ. If the distance d on G/K satisfies these conditions, then any lattice in G has property ($\underline{H}$) for the induced distance.

To illustrate these conditions, let us describe a fundamental example. Take $G = SL_n(\mathbf{R})$, $K = SO_n(\mathbf{R})$ and set, for $x, y \in G$:

$$d(x,y) = \log \|x^{-1}y\| + \log \|y^{-1}x\|$$

where $\| \ \|$ denotes the operator norm of an element acting on $\mathbf{R}^n$. This formula defines a right G-invariant pseudo-distance which is left K-invariant, and this induces a distance on G/K. Now, there exists a unique $a \in \overline{A^+}$ such that $x^{-1}y = KaK$, say

$$a = \begin{pmatrix} e^{\lambda_1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & e^{\lambda_n} \end{pmatrix}, \ \lambda_i \in \mathbf{R}, \ \sum_{i=1}^{n} \lambda_i = 0, \ \lambda_1 \geq \dots \geq \lambda_n$$

and

$$\begin{aligned} d(x,y) &= \lambda_1 - \lambda_n = \sum_{i=1}^{n-1} \alpha_i(\log a) \\ &= (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) + \dots + (\lambda_{n-1} - \lambda_n) = \lambda_1 - \lambda_n, \end{aligned}$$

where the maps $\alpha_i : \mathfrak{a} \to \mathbf{R}$ with

$$\alpha_i \begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n \end{pmatrix} = \lambda_i - \lambda_{i+1}$$

form a basis for Σ.

PROPOSITION 2.5. *If a discrete group Γ acts freely and by isometries on a discrete metric space (X, d) having both properties (L_δ) and (H_δ) for some $\delta \geq 0$, then it has property (RD).*

PROOF. The proof of this proposition is basically the proof of Proposition 2.3 in [**14**]. Let us consider the groupoid $\mathcal{G}$ given as follows:

$$\mathcal{G} = X \times X / \sim$$

where $(x,y) \sim (s,t)$ if there exists $\gamma \in \Gamma$ with $x = \gamma s, y = \gamma t$. We write $[x,y]$ for the class of (x,y) in $\mathcal{G}$, and

$$\mathcal{G}^0 = \{[x,y] \in \mathcal{G} | x = \gamma y \text{ for some } \gamma \in \Gamma\}$$

with source and range given by

$$\begin{aligned} s, r : \mathcal{G} &\to \mathcal{G}^0 \\ [x,y] &\mapsto s[x,y] = [y,y], r[x,y] = [x,x] \end{aligned}$$

so that the composable elements are

$$\mathcal{G}^2 = \{([x,y],[s,t]) \in \mathcal{G} \times \mathcal{G} | y = \gamma s \text{ for a } \gamma \in \Gamma\}$$

and $[x,y] \cdot [s,t] = [x,y] \cdot [\gamma s, \gamma t] = [x, \gamma t]$ if $y = \gamma s$. For $f, g \in \mathbf{R}_+\mathcal{G}$, the convolution is given by

$$f *_{\mathcal{G}} g[x,y] = \sum_{z \in X} f[x,z]g[z,y]. \qquad ([x,y] \in \mathcal{G})$$

It is enough to prove that there exists a polynomial P such that for every $r \in \mathbf{R}_+$, $f, g, h \in \mathbf{R}_+\mathcal{G}$ and $\operatorname{supp}(f) \subset \mathcal{G}_r = \{[x,y] \in \mathcal{G} | d(x,y) \leq r\}$, the following inequality holds:

$$f *_{\mathcal{G}} g *_{\mathcal{G}} h[x,x] \leq P(r)\|f\|_2\|g\|_2\|h\|_2 \tag{1}$$

for every $x \in X$, where $\|f\|_2^2 = \sum_{[x,y]\in\mathcal{G}} f[x,y]^2$. Indeed, from (1) we conclude that Γ has property (RD) by using Proposition 1.5 point (4) and defining for a fixed $x_0 \in X$, a linear map $T : \mathbf{C}\Gamma \to \mathbf{C}\mathcal{G}$ by

$$T(f)[x,y] = \begin{cases} f(\gamma) & \text{if } [x,y] = [x_0, \gamma x_0] \\ 0 & \text{otherwise .} \end{cases}$$

so that $T(f)[x_0,x_0] = f(e)$. One can check that $\|T(f)\|_2 = \|f\|_2$ and that $T(f * g) = T(f) *_{\mathcal{G}} T(g)$ for any $f, g \in \mathbf{C}\Gamma$, and hence T is an isometric embedding of normed algebras.

Notice that for $x_0 \in X$,

$$f *_{\mathcal{G}} g *_{\mathcal{G}} h[x_0,x_0] = \sum_{y,z \in X^2} f[x_0,y]g[y,z]h[z,x_0]$$

$$\leq \sum_{x \in \Gamma\backslash X} \sum_{y,z \in X^2} f[x,y]g[y,z]h[z,x] = \sum_{x,y,z \in \Gamma\backslash X^3} f[x,y]g[y,z]h[z,x]$$

and because of property (L_δ), we have that

$$\sum_{x,y,z \in \Gamma\backslash X^3} f[x,y]g[y,z]h[z,x] \leq \sum_{\substack{x,y,z,t \in \Gamma\backslash X^4 \\ xty, ytz, ztx \ \delta-\text{paths}}} f[x,y]g[y,z]h[z,x]$$

For xty a δ-path and $d(x,y) \leq r$, then $d(x,t) \leq r+\delta$ and $d(t,y) \leq r+\delta$. Define $H_1 = \ell^2\mathcal{G}$, $H_2 = \ell^2\mathcal{G}_{r+\delta} = H_3$ and $T_f \in \mathcal{L}(H_1, H_2)$ given as a matrix by

$$T_f([t,x],[v,y]) = \begin{cases} f[x,y] & \text{if } t \text{ is in the orbit of } v \text{ (so we assume } t = v\text{),} \\ & \text{and if } xty \text{ is a } \delta-\text{path} \\ 0 & \text{otherwise} \end{cases}$$

(T_f maps H_1 in H_2 because f is supported in a ball of radius r). In the same way we define $T_g \in \mathcal{L}(H_3, H_1)$ and $T_h \in \mathcal{L}(H_2, H_3)$. For $[t,x] \in \mathcal{G}$ we have

$$(T_f \circ T_g \circ T_h)([t,x],[t,x]) = \sum_{\substack{y,z \in X^2 \\ xty, ytz, ztx \ \delta-\text{paths}}} f[x,y]g[y,z]h[z,x]$$

and thus

$$\begin{aligned}\mathrm{Trace}(T_f \circ T_g \circ T_h) &= \sum_{[x,t]\in\mathcal{G}} (T_f \circ T_g \circ T_h)([t,x],[t,x]) \\ &= \sum_{x,t\in\Gamma\backslash X^2} \sum_{\substack{y,z\in X^2 \\ xty,ytz,ztx\ \delta-\text{paths}}} f[x,y]g[y,z]h[z,x] \\ &= \sum_{\substack{x,y,z,t\in\Gamma\backslash X^4 \\ xty,ytz,ztx\ \delta-\text{paths}}} f[x,y]g[y,z]h[z,x].\end{aligned}$$

Now we use that $\mathrm{Trace}(T_f \circ T_g \circ T_h) \leq \|T_f\|_{HS}\|T_g\|_{HS}\|T_h\|_{HS}$ and evaluate those Hilbert-Schmidt norms:

$$\begin{aligned}\|T_f\|^2_{HS} &= \sum_{[t,x][v,y]} |T_f([t,x],[v,y])|^2 = \sum_{\substack{x,y,t\in\Gamma\backslash X^3, d(x,t)\leq r+\delta \\ xty\ \delta-\text{path}}} |f[x,y]|^2 \\ &\leq P_\delta(r+\delta) \sum_{x,y\in\Gamma\backslash X^2} |f[x,y]|^2 = P_\delta(r+\delta)\|f\|_2^2,\end{aligned}$$

the last inequality holding because of property (H_δ). Similarly:

$$\begin{aligned}\|T_g\|^2_{HS} &= \sum_{\substack{[t,y][v,z] \\ \in\mathcal{G}_{r+\delta}\times\mathcal{G}}} |T_g([t,y],[v,z])|^2 = \sum_{\substack{y,z,t\in\Gamma\backslash X^3, d(t,y)\leq r+\delta \\ ytz\ \delta-\text{path}}} |g[x,y]|^2 \\ &\leq P_\delta(r+\delta) \sum_{y,z\in\Gamma\backslash X^2} |g[y,z]|^2 = P_\delta(r+\delta)\|g\|_2^2\end{aligned}$$

and

$$\begin{aligned}\|T_h\|^2_{HS} &= \sum_{\substack{[t,z][v,x] \\ \in\mathcal{G}_{r+\delta}\times\mathcal{G}_{r+\delta}}} |T_h([t,z],[v,x])|^2 = \sum_{\substack{z,x,t\in\Gamma\backslash X^3, d(t,x)\leq r+\delta \\ xtz\ \delta-\text{path}}} |h[z,x]|^2 \\ &\leq P_\delta(r+\delta) \sum_{z,x\in\Gamma\backslash X^2} |h[z,x]|^2 = P_\delta(r+\delta)\|h\|_2^2.\end{aligned}$$

□

Uniform nets in products of hyperbolic spaces

DEFINITION 2.6. Let (X,d) be a metric space, for any $r \in \mathbf{R}_+$ and $x \in X$, let $B(x,r) \subset X$ denote the open ball of radius r centered at x. A subset $Y \subset X$ is a *uniform net* in X if there exists two constants r_Y and R_Y in $\mathbf{R}_+$ such that

$$\begin{aligned} B(y,r_Y)\cap Y = \{y\} &\quad \text{for every } y \in Y \\ B(x,R_Y)\cap Y \neq \emptyset &\quad \text{for every } x \in X\end{aligned}$$

The metric space X is said to have *uniformly bounded geometry* if, for any uniform net Y in X and any $r \geq 0$, there exists a constant C (depending only on r and Y) such that, for every $x \in X$, the cardinality of $B(x,r)\cap Y$ is bounded by C.

REMARK 2.7. In a δ-hyperbolic space $(\mathcal{H}, d)$, property (L) holds. Indeed, let x, y, z be a triple of points in $\mathcal{H}$ and Δ be any geodesic triangle with vertices x, y, z. By definition there exist $u \in \gamma_{xy}, v \in \gamma_{yz}, w \in \gamma_{zx}$ so that $d(u, v), d(v, w), d(w, u) \leq \delta$. To prove property (L) it is enough to find some $\xi \in \mathcal{H}$ so that $x\xi y, y\xi z$ and $z\xi x$ are 2δ-paths. But any of the points u, v, w will do, let us check that with $\xi = u$. Since $u \in \gamma_{xy}$ is on a geodesic between x and y, there is nothing to do for the path xuy. Let us look at the path yuz:

$$\begin{aligned} d(y,u) + d(u,z) &\leq d(y,v) + d(v,u) + d(u,v) + d(v,z) \\ &\leq d(y,v) + 2\delta + d(v,z) = d(y,z) + 2\delta \end{aligned}$$

and similarly for the path zux.

LEMMA 2.8. *Let $(\mathcal{H}_1, d_1) \ldots (\mathcal{H}_n, d_n)$ be geodesic δ_i-hyperbolic spaces. Endow $\mathcal{K} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with the ℓ^1 combination of the distances d_i, and let $X \subset \mathcal{K}$ be a uniform net. Then X endowed with the induced metric of $\mathcal{K}$ has property (L).*

PROOF. Take $\delta \geq 2\left(\sum_{i=1}^n \delta_i + R_X\right)$ and $x, y, z \in X$, we have to prove that the triple x, y, z is δ-retractable. We write $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ where for each i, $x_i, y_i, z_i \in \mathcal{H}_i$. Because of the previous Remark, for every $i = 1, \ldots, n$, there exists $t'_i \in \mathcal{H}_i$ such that $x_1 t'_i y_i$, $y_i, t'_i z_i$ and $z_i t'_i x_i$ are $(2\delta_i)$-paths. Take $t' = (t'_1, \ldots, t'_n)$ and $t \in X$ such that $d(t, t') \leq R_X$ (such a t exists by uniformity), then the paths xty, ytz and ztx will be δ-paths. Indeed:

$$\begin{aligned} d(x,t) + d(t,y) &= \sum_{i=1}^n \left(d_i(x_i, t'_i) + d_i(t'_i, y_i)\right) + d(t', t) + d(t, t') \\ &\leq \sum_{i=1}^n \left(d_i(x_i, y_i) + 2\delta_i\right) + 2R_X \leq d(x,y) + \delta. \end{aligned}$$

and similarly for ytz and ztx. □

REMARK 2.9. It is an easy observation from the proof of this lemma that if a metric space $\mathcal{K}$ has property (L), then any uniform net Y will have property (L) as well (take $\delta' \geq \delta + 2R_Y$). Conversely the same statement holds, namely if $Y \subset \mathcal{K}$ is a uniform net and has property (L), then the ambient space $\mathcal{K}$ will also have property (L). Indeed, take $\delta' \geq \delta + 4R_Y$ and $x, y, z \in \mathcal{K}$, then there exists $x', y', z' \in Y$ at respective distances from x, y, z less than R_Y, and $t \in Y$ so that $x'ty', y'tz'$ and $z'tx'$ are δ-paths. We compute:

$$\begin{aligned} d(x,t) + d(t,y) &\leq d(x,x') + d(x',t) + d(t,y') + d(y',y) \\ &\leq 2R_Y + d(x',t) + d(t,y') \leq 2R_Y + \delta + d(x',y') \\ &\leq 2R_Y + \delta + d(x',x) + d(x,y) + d(y,y') \\ &\leq 4R_Y + \delta + d(x,y) \leq \delta' + d(x,y) \end{aligned}$$

and similarly for ytz and ztx.

We will now turn to property ($\underline{H}$), the following lemma is the analogue of the previous remark but for property ($\underline{H}$).

LEMMA 2.10. *Let X be a uniform net in a locally compact geodesic metric space $(\mathcal{K}, d)$. If X satisfies property ($\underline{H}$), then so will any other uniform net Y in $\mathcal{K}$ (but for some other polynomial).*

PROOF. Take $x, y \in \mathcal{K}$ and set, for any $\delta \geq 0$,

$$\Upsilon_{(\delta,r)}(x,y) = \{t \in \mathcal{K} | xty \text{ is a } \delta - \text{path}, \ d(x,t) \leq r\}.$$

Because of property ($\underline{H}$) we can cover $\Upsilon_{(\delta,r)}(x,y)$ with $P_\epsilon(r)$ balls of radius R_X centered at each point of $\Upsilon_{(\delta,r)} \cap X$, where P_ϵ denotes the polynomial associated to the constant $\epsilon \leq 2R_X + \delta$ in the definition of property ($\underline{H}$). Now, take $z \in Y$ so that xzy is a δ-path. Since in particular $z \in \mathcal{K}$, we can find $t \in X$ at a distance less than R_X, and thus z is in the ball of radius R_X centered at t. It is now obvious that xty is a $2R_X + \delta$-path. But in each ball of radius R_X there is a uniformly bounded number N of elements of Y, so that

$$\begin{aligned} & \sharp\{t \in Y \text{ such that } d(x,t) \leq r, \text{ and } xty \ \delta - \text{path}\} \\ \leq \ & N\sharp\{t \in X \text{ such that } d(x,t) \leq r, \text{ and } xty \ (2R_X + \delta) - \text{path}\} \\ \leq \ & NP_\epsilon(r) \end{aligned}$$

and thus Y satisfies property ($\underline{H}$), choosing $P_\delta = NP_\epsilon$. □

REMARK 2.11. This lemma shows that property ($\underline{H}$) can be viewed as a property of the ambient space

Before proceeding, let us make some remarks about δ_0-paths in a δ-hyperbolic space $(\mathcal{H}, d)$ and estimate the degree of the polynomials involved for property ($\underline{H}$).

REMARK 2.12. Take $x, y \in \mathcal{H}$ and $t \in \Upsilon_{(\delta_0,r)}(x,y)$. Remember that we defined

$$\Upsilon_{(\delta_0,r)}(x,y) = \{t \in \mathcal{H} | xty \text{ is a } \delta_0 - \text{path}, d(x,t) \leq r\}$$

(1) Such a t will be at a uniformly bounded distance of any geodesic between x and y. Indeed, let $\gamma_{xy}, \gamma_{xt}, \gamma_{yt}$ be a geodesic triangle with vertices x, y, t. By definition, there exists $u \in \gamma_{yt}, v \in \gamma_{xt}, w \in \gamma_{xy}$ with diameter less than δ, so that

$$d(t, \gamma_{xy}) \leq d(t,w) \leq d(t,v) + d(v,w) \leq d(t,v) + \delta$$

and since $u \in \gamma_{yt}, v \in \gamma_{xt}$ we have that

$$d(x,v) + d(v,t) + d(t,u) + d(u,y) \ = \ d(x,t) + d(t,y) \leq d(x,y) + \delta_0.$$

Combined with

$$\begin{aligned} d(x,v) + 2\delta + d(u,y) + \delta_0 \ & \geq \ d(x,v) + d(v,w) + d(w,u) + d(u,y) + \delta_0 \\ & \geq \ d(x,y) + \delta_0 \\ & \geq \ d(x,v) + d(v,t) + d(t,u) + d(u,y) \end{aligned}$$

we get that

$$2\delta + \delta_0 \geq d(v,t) + d(t,u),$$

i. e. $d(t,\gamma_{xy}) \leq 3\delta + \delta_0$. Notice that this is not true in a Euclidean space, the distance of t to the geodesic γ_{xy} might be growing like the square root of the distance between x and y.

(2) Let us estimate how many balls of radius ρ (depending just on δ and δ_0) are needed to cover $\Upsilon_{(\delta_0,r)}(x,y)$. We can cover $\Upsilon_{(\delta_0,r)}(x,y)$ with $|r|$ balls of radius $\rho = 3\delta + \delta_0 + 1/2$, where $|r|$ denotes the integer part of r. Indeed, denote by γ_{xy} a geodesic between x and y, and $z_0 = x, z_1, \ldots, z_{|r|} = z$ points at distances 1 on γ_{xy} (so that $1 \leq d(x,z) \leq r$). Take $t \in \Upsilon_{(\delta_0,r)}(x,y)$, then by the first part of this remark, $d(t,\gamma_{xy}) \leq 3\delta + \delta_0$, that is, there exists $k \in \{1,\ldots,|r|\}$ such that $d(z_k,t) \leq 3\delta + \delta_0 + 1/2$, and thus $t \in B(z_k,\rho)$.

Lemma 2.13. *Let $(\mathcal{H},d)$ be a complete, locally compact δ-hyperbolic space of uniformly bounded geometry and $X \subset \mathcal{H}$ be a uniform net. Then X has property $(\underline{H})$.*

Proof. In view of the last point of the previous remark, we deduce that the polynomial needed for property $(\underline{H})$ for a uniform net X in $\mathcal{H}$ is given by $P_{\delta_0}(r) = N_{\delta_0} r$, where N_{δ_0} denotes the cardinality of points of X lying in a ball of radius $3\delta + \delta_0 + 1/2$. This N_{δ_0} exists and is finite because X is discrete in $\mathcal{H}$ and we assumed $\mathcal{H}$ to be geodesic, complete, locally compact and of and of uniformly bounded geometry, which implies that closed balls are compact and that the cardinality of points of X lying in a ball of fixed radius is uniformly bounded. □

Lemma 2.14. *Let $(\mathcal{K}_1,d_1)\ldots(\mathcal{K}_n,d_n)$ be metric spaces whose uniform nets all have property $(\underline{H})$. Endow $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$ with the ℓ^1 combination of the distances d_i, then any uniform net in $\mathcal{K}$ has property $(\underline{H})$.*

Proof. Because of Lemma 2.10, it is enough to show that one particular uniform net has property $(\underline{H})$. To do that, let $X_1 \subset \mathcal{K}_1 \ldots X_n \subset \mathcal{K}_n$ be uniform nets and look at $X = X_1 \times \cdots \times X_n$, which is a uniform net in $\mathcal{K}$. Take $x,y \in X$, $\delta, r \geq 0$ and $t \in \Upsilon_{(\delta,r)}(x,y)$. We write x,y,t in coordinates, that is to say $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n)$ and $t = (t_1,\ldots,t_n)$, where $x_i,y_i,t_i \in X_i$ for any $i = 1,\ldots,n$. On each factor X_i, $x_it_iy_i$ will be a δ-path as well, and since we are considering the ℓ^1 combination of distances, $d_i(x_i,t_i) \leq r$. On each X_i there is by assumption at most $P_{\delta,i}(r)$ of those points t_i and thus on X we have at most

$$P_\delta(r) = \prod_{i=1}^{n} P_{\delta,i}(r)$$

t's at distance to x less than r and such that xty is a δ-path. Obviously P_δ is again a polynomial (of degree the sum of the degrees of the $P_{\delta,i}$'s) and thus X has property $(\underline{H})$. □

REMARK 2.15. If a group Γ acts by isometries and with uniformly bounded stabilizers on a discrete metric space (X, d) having both properties (L) and $(\underline{H})$, then it will act freely and by isometries on a discrete metric space (X', d') having also properties (L) and $(\underline{H})$. Let us show how such an X' can be built:

Take $(x_i)_{i\in J}$ a (possibly infinite) sequence of points in X so that $X = \coprod_{i\in J} \Gamma x_i$, where J is some countable indexing set (it is the partition of X in Γ-orbits). For any $i \in J$, the stabilizer of x_i (and thus of any of the γx_i for $\gamma \in \Gamma$) is a finite subgroup of Γ. Set

$$X' = \coprod_{i\in J} \Gamma$$

and take the diagonal action of Γ on X'. Set $\theta : X' \to X$ the obvious orbit map, and define a distance on X' as follows:

$$d'(x, y) = \theta^*(d)(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 + d(\theta(x), \theta(y)) & \text{if } x \neq y \end{cases}$$

Since the stabilizers are uniformly bounded, X is a uniform net in X', thus applying Remark 2.9 and Lemma 2.10 we can deduce that X' has both properties (L) and $(\underline{H})$ as well.

COROLLARY 2.16. *Any cocompact lattice in the isometry group of a finite product of hyperbolic spaces has property (RD).*

PROOF. Let $\mathcal{K}$ be a finite product of hyperbolic spaces, and $\Gamma < \mathrm{Iso}(\mathcal{K})$ a cocompact lattice. Take any $x_0 \in \mathcal{K}$ then $X = \Gamma x_0$ its obit in $\mathcal{K}$ is a uniform net, and has both properties $(\underline{H})$ and (L) because of Lemmas 2.8 and 2.14. Now, Γ acts by isometries and with uniformly bounded stabilizers on the metric space X with the induced metric of $\mathcal{K}$. But using the previous remark we can assume that the action is free and apply Proposition 2.5 to conclude that Γ has property (RD). □

Coxeter groups

In this section we will see that any finite product of trees has both properties (L_0) and (H_0). There are neither regularity nor finiteness assumptions on the degrees of the vertices of the trees. This case was not included in the previous section since we had the assumption that the hyperbolic spaces are locally compact. Deducing property (RD) for Coxeter groups is a straightforward application of a result due to T. Januszkiewicz, see [**11**], in which he shows that Coxeter groups act on products of finitely many trees. This remark has been done by N. Higson during a workshop on Non-Positive Curvature in Penn State

University while I was presenting the first two sections of this current Chapter.

LEMMA 2.17. *Any finite product of trees, endowed with the graph theoretical metric associated to the 1-skeleton of the product, has both properties* (L_0) *and* (H_0).

PROOF. Denote by X the set of vertices of a finite product of trees. That X has property (L_0) follows from the proof of Lemma 2.8. That X has property (H_0) follows from the fact that for any two points $x, y \in X$ there exists an apartment containing their convex hull. Since an apartment in this context is a copy of $\mathbf{R}^n$ endowed with the ℓ^1-metric, property (H_0) holds as well. □

REMARK 2.18. The free group $\mathbf{F}_\infty$ on infinitely many generators has property (RD) with respect to its word length since it acts freely on its Cayley graph which is a regular tree (not locally finite). Notice that the word length on $\mathbf{F}_\infty$ is not a proper length.

DEFINITION 2.19. A *Coxeter group* is a discrete group Γ given by the presentation with a finite set of generators $W = \{w_1, \dots, w_n\}$ and a finite set of relations reading as follows:

$$w_i^2 = 1 = (w_i w_j)^{m_{ij}}$$

where m_{ij} is either ∞ (and then there is no relation between w_i and w_j) or an integer greater or equal 2.

COROLLARY 2.20. *Coxeter groups have property (RD).*

PROOF. Let Γ be a Coxeter group. By Lemmas 2 and 3 in [**11**], Γ acts properly and simplicially on a finite product of trees. Coxeter groups being linear, we can apply Selberg's Lemma, so Γ has a normal torsion free subgroup Γ_0 of finite index and thus the action has uniformly bounded stabilizers. Combining Lemma 2.17 with Proposition 2.5 we deduce property (RD) for Γ. □

CHAPTER 3

Triples of points in $SL_3(\mathbf{H})$ and $E_{6(-26)}$

In this chapter we will answer a question posed by V. Lafforgue in [**14**], which was to know if his two lemmas concerning triples of points in $SL_3(\mathbf{R})/SO_3(\mathbf{R})$ and $SL_3(\mathbf{C})/SU_3(\mathbf{C})$ still hold for triples in $SL_3(\mathbf{H})/SU_3(\mathbf{H})$ or $E_{6(-26)}/F_{4(-52)}$. Observing that these lemmas are in fact "three points conditions" it will be enough to prove that if X denotes $SL_3(\mathbf{H})/SU_3(\mathbf{H})$ or $E_{6(-26)}/F_{4(-52)}$ (endowed with some Finsler distance), then for any three points in X there exists a totally geodesic embedding of $SL_3(\mathbf{C})/SU_3(\mathbf{C})$ (again endowed with some Finsler distance) containing those three points. By *totally geodesic embedding*, we mean that for any two points in the image of the embedding, then any geodesic between those two points is also in the image of the embedding. The first two sections will be devoted to explaining those embeddings, and the last one to explain how we answer V. Lafforgue's question.

The case of $SL_3(\mathbf{H})$

We will write $\mathbf{H}$ for the *Hamilton's Quaternion algebra*, which is a 4 dimensional algebra over $\mathbf{R}$, whose basis is given by the elements $1, i, j$ and k, satisfying

$$i^2 = j^2 = k^2 = -1\,,\ ij = k\,,\ ki = j\,,\ jk = i.$$

This is an associative division algebra endowed with an involution $h \mapsto \overline{h}$ which is the identity over 1, and minus the identity over i, j and k. A norm on $\mathbf{H}$ can be given by $|h| = \sqrt{h\overline{h}} \in \mathbf{R}_+$ (we identify $\mathbf{R}$ with $\mathbf{R} \cdot 1$ in $\mathbf{H}$). A quaternion will be called a *unit* if of norm one, *real* if lying in span$\{1\}$ and *imaginary* if lying in span$\{1\}^\perp$ (for the scalar product of $\mathbf{R}^4$ which turns the above described basis in an orthonormal basis). Note that an imaginary unit has square -1.

LEMMA 3.1. *Any element $h \in \mathbf{H}$ is contained in a commutative subfield of $\mathbf{H}$.*

PROOF. Take $h \in \mathbf{H}$, if h is real, then $h \in$ span$\{1\} \simeq \mathbf{R}$. Otherwise, define

$$i(h) = \frac{h - \overline{h}}{|h - \overline{h}|}.$$

The element $i(h)$ is a purely imaginary unit and $\operatorname{span}\{1, i(h)\}$ is a commutative field (thus isomorphic to $\mathbf{C}$ because of $\mathbf{R}$-dimension 2). □

REMARK 3.2. Because of $i(h)$ being a unit, the map $\mathbf{C} \to \mathbf{H}$ sending 1 to 1 and i to $i(h)$ is an isometry.

DEFINITION 3.3. Denote by I the identity matrix in $M_3(\mathbf{C})$, and

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_6(\mathbf{C}),$$

we write

$$\begin{aligned} M_3(\mathbf{H}) &= \{M \in M_6(\mathbf{C}) \text{ such that } JMJ^{-1} = \overline{M}\} \\ SL_3(\mathbf{H}) &= \{M \in M_3(\mathbf{H}) \text{ such that } \det(M) = 1\} \\ SU_3(\mathbf{H}) &= \{M \in SL_3(\mathbf{H}) \text{ such that } MM^* = M^*M = I\}. \end{aligned}$$

REMARKS 3.4. 1) Given any imaginary unit $\mu \in \mathbf{H}$, we can write $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}\mu$, and thus decompose any $h \in \mathbf{H}$ as $h = h_1 + h_2\mu$, with $h_1, h_2 \in \mathbf{C}$. Similarly, any 3×3 matrix M with coefficients in $\mathbf{H}$ can be written $M = M_1 + M_2\mu$, with $M_1, M_2 \in M_3(\mathbf{C})$. The homomorphism

$$Z : M \mapsto \begin{pmatrix} M_1 & -\overline{M_2} \\ M_2 & \overline{M_1} \end{pmatrix}$$

gives then an isomorphism between the algebra of 3×3 matrices with coefficients in $\mathbf{H}$ (and usual multiplication) and $M_3(\mathbf{H})$ as just defined.

Let us define a "scalar product" on $\mathbf{H}^3$ with values in $\mathbf{H}$, i.e. a map $\langle\ ,\ \rangle : \mathbf{H}^3 \times \mathbf{H}^3 \to \mathbf{H}$ by

$$\langle v, w \rangle = \sum_{i=1}^{3} \overline{v}_i w_i \qquad (v, w \in \mathbf{H}^3).$$

This is a bilinear map satisfying, for any $v, w \in \mathbf{H}^3$: $\langle v\lambda, w \rangle = \overline{\lambda}\,\langle v, w \rangle$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$ and $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$. Moreover, if for $M = (m_{ij})$ a 3×3 matrix with coefficients in $\mathbf{H}$ we set $M^* = (\overline{m}_{ji})$, then

$$\langle Mv, w \rangle = \langle v, M^* w \rangle \qquad (v, w \in \mathbf{H}^3),$$

and $Z(M^*) = Z(M)^*$, where $Z(M)^*$ is the adjoint of $Z(M)$ for the usual scalar product on $\mathbf{C}^6$. This shows that Z is in fact a *-isomorphism.

2) It is a direct computation to see that elements of type

$$\mathcal{T}(\lambda, \mu, \eta) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix}$$

with λ, μ, η units in $\mathbf{H}$ actually belong to $SU_3(\mathbf{H})$. We will write $\mathcal{T}_\lambda$ for $\mathcal{T}(\lambda, 1, 1)$.

DEFINITION 3.5. Let $\mathbf{K}$ denote $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. We set

$$X_{\mathbf{K}} = SL_3(\mathbf{K})/SU_3(\mathbf{K}).$$

It will be convenient for us to choose

$$\{M \in SL_3(\mathbf{K}) \text{ such that } M^* = M, M \text{ positive}\}$$

as a model for $X_{\mathbf{K}}$. On $X_{\mathbf{K}}$ we consider the action of $SL_3(\mathbf{K})$ given by:

$$\begin{array}{rcl} SL_3(\mathbf{K}) \times X_{\mathbf{K}} & \to & X_{\mathbf{K}} \\ (\, g \;,\; z \,) & \mapsto & g(z) = (gz^2g^*)^{1/2}. \end{array}$$

The action is transitive since for $\in X_{\mathbf{K}}$, setting $g = M$ we get that $g(I) = M$. We equip $X_{\mathbf{K}}$ with the distance

$$d_{\mathbf{K}}(x, y) = \log \|x^{-1}y\| + \log \|y^{-1}x\|.$$

where $\| \ \|$ denotes the operator norm on $SL_3(\mathbf{K})$ acting on $\mathbf{K}^3$. Notice that for $a \in X_{\mathbf{K}}$ a diagonal matrix (thus real), if we assume that

$$a = \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} \text{ with } \alpha_1 \geq \alpha_2 \geq \alpha_3 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 0,$$

we get that $d_{\mathbf{K}}(a, I) = \alpha_1 - \alpha_3$. Denote by A the set of all diagonal matrices in $X_{\mathbf{K}}$, and by

$$\overline{A^+} = \{ \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} \in A \text{ with } \alpha_1 \geq \alpha_2 \geq \alpha_3 \}$$

Because of $X_{\mathbf{K}}$ consisting of hermitian matrices, A only contains real matrices, hence does not depend on $\mathbf{K}$, and thus we do not index A with $\mathbf{K}$. To stick with the classical terminology, we will call *flats* the sets of type $g(A)$ for a $g \in SL_3(\mathbf{K})$. Those are the maximal flat subspaces of the symmetric space associated to $SL_3(\mathbf{K})$, that can be viewed as $X_{\mathbf{K}}$ endowed with its Riemannian distance. In case where $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$, this distance is the one used by V. Lafforgue in [**14**]. See the work of P. Planche in [**22**] for the proof that this distance is indeed a Finsler distance.

REMARK 3.6. Let us now show that $SL_3(\mathbf{K})$ acts on $X_{\mathbf{K}}$ by isometries with respect to the above given distance (in other words, the distance is $SL_3(\mathbf{K})$-invariant). For $x, y \in X_{\mathbf{K}}$, we have to show that the operator norm of $x^{-1}y$ is equal to that of $z = g(x)^{-1}g(y) = (gx^2g^*)^{-1/2}(gy^2g^*)^{1/2}$. But for any $k_1, k_2 \in SU_3(\mathbf{K})$, $\|z\| = \|k_1zk_2\|$, so setting $k_1 = (gx)^{-1}(gx^2g^*)^{1/2}$ and $k_2 = (gy^2g^*)^{-1/2}gy$, we get that $k_1zk_2 = x^{-1}y$, and $k_1, k_2 \in SU_3(\mathbf{K})$.

Notice that for $g \in SL_3(\mathbf{K})$, the standard action $z \mapsto gzg*$ is not isometric. Indeed, the operator norm is in general not invariant by conjugation by g unless $g = k \in SU_3(\mathbf{K})$, and in that case $k(z) = (kz^2k^*)^{1/2} = (kzk^*kzk^*)^{1/2} = kzk^*$.

LEMMA 3.7. *With the above given notations, when $\mathbf{K} = \mathbf{C}$, the distance is such that every geodesic between two points x and y in $X_{\mathbf{C}}$ is contained in any flat containing x and y.*

PROOF. We can assume that $x = I$ and that $y \in \overline{A^+}$. Take $z \in X_{\mathbf{C}}$ on a geodesic between x and y, that is, $d(I, z) + d(z, y) = d(I, y)$. We will first show that z has to be diagonal. Since $SL_3(\mathbf{C})$ acts transitively on $X_{\mathbf{C}}$, we can write

$$y = \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix}, z = k(\begin{pmatrix} e^{\beta_1} & 0 & 0 \\ 0 & e^{\beta_2} & 0 \\ 0 & 0 & e^{\beta_3} \end{pmatrix}), z^{-1}y = k'(\begin{pmatrix} e^{\gamma_1} & 0 & 0 \\ 0 & e^{\gamma_2} & 0 \\ 0 & 0 & e^{\gamma_3} \end{pmatrix})$$

with $k, k' \in SU_3(\mathbf{C})$, $\beta_1 \geq \beta_2 \geq \beta_3$ and $\gamma_1 \geq \gamma_2 \geq \gamma_3$, so that the assumption on z amounts to

$$\alpha_1 - \alpha_3 = \beta_1 - \beta_3 + \gamma_1 - \gamma_3. \qquad (*)$$

Since $y = z(z^{-1}y)$, it implies that $\alpha_1 \leq \beta_1 + \gamma_1$ (because $e^{\alpha_1} = \|y\| \leq \|z\|\|z^{-1}y\| = e^{\beta_1}e^{\gamma_1}$). We claim that $\alpha_1 = \beta_1 + \gamma_1$. Indeed, assume that $\alpha_1 < \beta_1 + \gamma_1$. After plugging this inequality in the equality $(*)$, it implies that $-\alpha_3 > -\beta_3 - \gamma_3$. This is a contradiction since

$$e^{-\alpha_3} = \|y^{-1}\| \leq \|y^{-1}z\|\|z^{-1}\| = e^{-\gamma_3}e^{-\beta_3}.$$

We then deduce that $\alpha_1 = \beta_1 + \gamma_1$, and $\alpha_3 = \beta_3 + \gamma_3$. But this means that the eigenvector of z associated to the eigenvalue e^{β_1} is equal to the one of $z^{-1}y$ associated to the eigenvalue e^{γ_1}, and similarly, the eigenvector of z associated to e^{β_3} is equal to the one of $z^{-1}y$ associated to e^{γ_3}, so that the last eigenvectors have to be the same as well. Since all eigenvectors agree, z commutes to $z^{-1}y$, and thus x, y, z all commute, i.e they are simultaneously orthogonally diagonalizable, so that we can choose $k = k'$. Now, if $\alpha_1 > \alpha_2 > \alpha_3$, the flat A is the only one containing both x and y, so that $k = k' = I$ and z has to be diagonal as well. If say $\alpha_1 = \alpha_2$, then

$$k = k' = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \text{ with } m \in SU_2(\mathbf{C}).$$

But since $\alpha_2 = \beta_2 + \gamma_2$, we deduce that $\beta_1 = \beta_2$ (indeed, $\beta_1 > \beta_2$ would imply $\alpha_1 = \beta_1 + \gamma_1 > \beta_2 + \gamma_2 = \alpha_2$). In particular, this means that for such a y, there is a unique finsler geodesic to I. □

LEMMA 3.8. *Let $\varphi : \mathbf{C} \to \mathbf{H}$ be an isometric injective ring homomorphism. It induces a group homomorphism*

$$\widetilde{\varphi} : SL_3(\mathbf{C}) \;\to\; SL_3(\mathbf{H})$$

which induces an isometric embedding

$$\overline{\varphi} : X_{\mathbf{C}} \;\to\; X_{\mathbf{H}}$$

PROOF. That $\widetilde{\varphi}$ is a homomorphism is clear in view of Remark 3.4. Since $\widetilde{\varphi}(x)^* = \widetilde{\varphi}(x^*)$, the map $\overline{\varphi}$ is well defined, so let us now show that it is an isometry. Since $SL_3(\mathbf{C})$ acts transitively and by isometries on $X_{\mathbf{C}}$, it will be enough to show that

$$d_{\mathbf{H}}(\overline{\varphi}(x), I) = d_{\mathbf{C}}(x, I)$$

for any $x \in X_{\mathbf{C}}$. Noticing that $\overline{\varphi}$ is the restriction of $\widetilde{\varphi}$ to $X_{\mathbf{C}}$, we deduce that for any $g \in SL_3(\mathbf{C})$, $\overline{\varphi}\,(g(x)) = \widetilde{\varphi}(g)\,(\overline{\varphi}(x))$. Now, for $x \in X_{\mathbf{C}}$, there exists $k \in SU_3(\mathbf{C})$ such that $k(x) = a$ is diagonal with positive coefficients. Since φ is a field homomorphism, it is the identity over the real numbers and so:

$$\begin{aligned} d_{\mathbf{C}}(x, I) &= d_{\mathbf{C}}(k(x), k(I)) = d_{\mathbf{C}}(a, I) = d_{\mathbf{H}}(a, I) \\ &= d_{\mathbf{H}}(\widetilde{\varphi}(k^*)(a), \widetilde{\varphi}(k^*)(I)) = d_{\mathbf{H}}(\overline{\varphi}(k^*(a)), I) \\ &= d_{\mathbf{H}}(\overline{\varphi}(x), I). \end{aligned}$$

□

REMARK 3.9. Any element of $X_{\mathbf{H}}$ can be diagonalized using elements of type $\mathcal{T}_h$ (described in Remark 3.4) and of $SO_3(\mathbf{R})$. Indeed, take $z = (z_{ij}) \in X_{\mathbf{H}}$, then $z' = \mathcal{T}_h(z)$ with $h = \overline{z_{12}}/|z_{12}|$ verifies that $z'_{ij} \in \mathbf{R}$ for $i, j = 1, 2$ (in case $z_{12} = 0$, or if it lies in $\mathbf{R}$ we can just skip this step). Take $k \in SO_2(\mathbf{R})$ diagonalizing the 2×2 matrix (z'_{ij}) $i, j = 1, 2$. Then

$$\begin{aligned} z'' &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} z' \begin{pmatrix} k^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} z''_{11} & z''_{12} & 0 \\ \overline{z''_{12}} & z''_{22} & z''_{23} \\ 0 & \overline{z''}_{23} & z''_{33} \end{pmatrix} \end{aligned}$$

with $z''_{11}, z''_{22}, z''_{33} \in \mathbf{R}$. Now set $h = \overline{z''_{12}}/|z''_{12}|$, and $\ell = z''_{23}/|z''_{23}|$, then $y = \mathcal{T}(h, 1, \ell)(z'')$ lies in $SL_3(\mathbf{R})$, and thus we can choose $m \in SO_3(\mathbf{R})$ diagonalizing it. Notice that we used that

$$\mathcal{T}(h, 1, \ell) = \mathcal{T}_h \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathcal{T}_\ell \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The above given argument shows that the subgroup of $SL_3(\mathbf{H})$ generated by the $\mathcal{T}_h$'s and $SL_3(\mathbf{R})$ acts transitively on $X_{\mathbf{H}}$. Since the $\mathcal{T}_h$'s and $SO_3(\mathbf{R})$ do stabilize I, we can conclude that $SL_3(\mathbf{H})$ is generated by elements of type $\mathcal{T}_h$ and $SL_3(\mathbf{R})$.

PROPOSITION 3.10. *For any three points in $X_{\mathbf{H}}$, there is a totally geodesic embedding of $X_{\mathbf{C}}$ containing those three points.*

PROOF. Let $x, y, z \in X_{\mathbf{H}}$. Up to multiplication by an element of $SL_3(\mathbf{H})$ (which is an isometry), we can assume that $x = I$ and y is a diagonal matrix (see the previous Remark). We write

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ \overline{z_{12}} & z_{22} & z_{23} \\ \overline{z_{13}} & \overline{z_{23}} & z_{33} \end{pmatrix}$$

with $z_{11}, z_{22}, z_{33} \in \mathbf{R}$ and z_{12}, z_{13} and z_{23} in $\mathbf{H}$. Consider the element

$$k = \begin{pmatrix} \overline{z_{12}}/|z_{12}| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{23}/|z_{23}| \end{pmatrix} \in SU_3(\mathbf{H}),$$

then $k(z)$ only has real elements except for $k(z)_{13} = \overline{k(z)_{31}}$, and $k(a) = a$ for any diagonal element a. Applying Lemma 3.1 to $k(z)_{13}$, we get an embedding of $\mathbf{C}$ in $\mathbf{H}$ which contains $k(z)_{13}$, and thus an isometrical embedding $\overline{\varphi} : X_{\mathbf{C}} \to X_{\mathbf{H}}$ whose image contains x, y and z.

Let us now prove that the embedding is totally geodesic. Take x, y in the image of $\overline{\varphi}$, so that $x = \overline{\varphi}(x')$ and $y = \overline{\varphi}(y')$ for $x', y' \in X_{\mathbf{C}}$. There is a $g \in SL_3(\mathbf{C})$ so that $x' = g(I)$ and $y' = g(a)$ for $a \in A$. Then $x = \overline{\varphi}(g(I)) = \widetilde{\varphi}(g)(I)$ and $y = \overline{\varphi}(g(a)) = \widetilde{\varphi}(g)(a)$. Since $\overline{\varphi}$ is the identity on A, it will obviously in the image of $\overline{\varphi}$, and thus so will $\widetilde{\varphi}(g)(A)$ be (that is, the flat $\widetilde{\varphi}(g)(A)$, which contains x and y is in the image of the embedding).

Finally, we claim that a geodesic γ between x and y lies in any flat containing them. Indeed, without loss of generality, we can assume that $x = I$ and $y \in A$. Take z on γ, by the first part of this proof, we can assume that $z \in X_{\mathbf{C}}$ and use Lemma 3.7 to conclude that z lies in any flat of $X_{\mathbf{C}}$ containing x and y, and thus γ will also lie in a flat of $X_{\mathbf{H}}$ containing both x and y. We conclude that $\gamma \subset \widetilde{\varphi}(g)(A) \subset \overline{\varphi}(X_{\mathbf{C}})$. □

The case of $E_{6(-26)}$

We will write $\mathbf{O}$ for the 8 dimensional non-associative algebra over $\mathbf{R}$, whose basis is given by the elements $e_0, e_1, \ldots, e_7$, satisfying

$$\begin{aligned} e_i e_0 = e_0 e_i = e_i, e_i^2 = -e_0 \quad & \text{for all } i = 1, \ldots, 7 \\ e_i e_j = -e_j e_i \quad & \text{if } i \neq j \text{ and } i, j \neq 0 \\ e_2 e_6 = e_3 e_4 = e_5 e_7 = e_1 \quad & \end{aligned}$$

and all those one can deduce by cyclically permuting the indices from 1 to 7. This is a non-associative division algebra, endowed with an involution $x \mapsto \overline{x}$ which is the identity over e_0, minus the identity over e_i, for all $i = 1, \ldots, 7$ and satisfies $\overline{xy} = \overline{y}\,\overline{x}$. A norm on $\mathbf{O}$ can be given by $|x| = \sqrt{x\overline{x}} \in \mathbf{R}_+$. The division algebra $\mathbf{O}$ is called the *Cayley Octonions*. An octonion will be called a *unit* if of norm one, *real* if lying in $\operatorname{span}\{e_0\}$ and *imaginary* if lying in $\operatorname{span}\{e_0\}^\perp$ (for the scalar product of $\mathbf{R}^8$ which turns the above described basis in an orthonormal basis).

The following theorem will allow us to apply the arguments used in the previous section to $E_{6(-26)}$:

THEOREM 3.11 (Artin, see [**26**] page 29). *Any two elements of* $\mathbf{O}$ *are contained in an associative subfield of* $\mathbf{O}$.

We will now give a definition of $E_{6(-26)}$, of a maximal compact subgroup $F_{4(-52)}$, and of a model for $X_{\mathbf{O}} = E_{6(-26)}/F_{4(-52)}$. This part is based on the work of H. Freudenthal, see [**3**] and [**4**] and has been explained to me by D. Allcock.

DEFINITION 3.12. Denote by $M_3(\mathbf{O})$ the set of 3×3 matrices with coefficients on $\mathbf{O}$. For $M = (m_{ij}) \in M_3(\mathbf{O})$, write $M^* = \overline{M}^t = (\overline{m}_{ji})$. The *exceptional Jordan algebra* over $\mathbf{R}$ is given by

$$\mathcal{J} = \{M \in M_3(\mathbf{O}) \text{ such that } M = M^*\}$$

which is stable under the *Jordan multiplication* given by

$$M \star N = \frac{1}{2}(MN + NM).$$

This composition law $\star$ is non-associative, but commutative. H. Freudenthal defined an application

$$\begin{array}{rcl} \det : \mathcal{J} & \to & \mathbf{R} \\ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} & \mapsto & \xi_1\xi_2\xi_3 - (\sum_{i=1}^{3} \xi_i |x_i|^2) + 2\Re(x_1x_2x_3) \end{array}$$

(where $\Re$ denotes the real part, and is well defined even without parenthesis), showed that

$$E_{6(-26)} = \{g \in GL(\mathcal{J}) \text{ such that } \det \circ g = \det\}$$

is a connected simple Lie group of real rank two, of dimension 78, Dynkin diagram of type E_6, Cartan index[1] equal to -26, and that

$$F_{4(-52)} = \{g \in E_{6(-26)} \text{ such that } g(I) = I\}$$

is the automorphism group of $\mathcal{J}$ and is a maximal compact subgroup in $E_{6(-26)}$, of dimension 52 and Dynkin diagram of type F_4. We refer to [**27**] for properties and formulas related to the map det and the law $\star$. Notice that in case we restrict the composition law $\star$ and the map det to elements in $\mathcal{J}$ with real coordinates we obtain the usual matrix multiplication and determinant form on $M_3(\mathbf{R})$. We finally define

$$\mathcal{D} = \{M \in \mathcal{J} \text{ such that } M = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}\},$$

[1]the Cartan index of a Lie group G is defined as $\dim(\mathfrak{p})$-$\dim(\mathfrak{k})$ where, in the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{p}$ is the eigenspace associated to the eigenvalue 1 for the Cartan involution, and $\mathfrak{k}$ the eigenspace associated to the eigenvalue -1.

those are the *diagonal elements* of $\mathcal{J}$. Notice that by definition of $\mathcal{J}$, the matrices in $\mathcal{D}$ must be real.

REMARKS 3.13. We will now explicitly show some elements of $E_{6(-26)}$ (see [**4**] for the proofs of their belonging to $E_{6(-26)}$):

1) Any element x of $SL_3(\mathbf{R})$ gives a map $x : \mathcal{J} \to \mathcal{J}$ by $M \mapsto x(M) = xMx^t$ which preserves the determinant.

2) Let a be a unit in $\mathbf{O}$, define

$$\varphi_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \overline{a} \end{pmatrix}.$$

The map $\psi_a : \mathcal{J} \to \mathcal{J}$, defined as $\psi_a(M) = \varphi_{\overline{a}} M \varphi_a$ (and no parenthesis are needed) is in $E_{6(-26)}$, and even in $F_{4(-52)}$.

3) H. Freudenthal proves (see [**3**], page 40) that elements in $\mathcal{J}$ are diagonalizable by elements in $F_{4(-52)}$ (meaning that for any $M \in \mathcal{J}$, there exists $k \in F_{4(-52)}$ such that $k(M)$ belongs to $\mathcal{D}$), that the elements on the diagonal are uniquely determined up to permutation, and characterize the equivalence class.

DEFINITION 3.14. We say that an element $M \in \mathcal{J}$ is *positive* if after diagonalization it only has positive elements. We define

$$X_{\mathbf{O}} = \{M \in \mathcal{J} \text{ such that } \det(M) = 1 \text{ and positive}\}.$$

For M positive, there exists $k \in F_{4(-52)}$ such that $k(M)$ is diagonal and only has positive elements, so that $d = \sqrt{k(M)}$ (in the real sense) is well defined, and $d \star d = k(M)$. We will write $\sqrt{M}$ for the element $k^{-1}(d)$ since it is well defined and

$$\sqrt{M} \star \sqrt{M} = k^{-1}(d) \star k^{-1}(d) = k^{-1}(d \star d) = k^{-1}k(M) = M.$$

We let $E_{6(-26)}$ act on $X_{\mathbf{O}}$ as follows:

$$\begin{array}{rcl} E_{6(-26)} \times X_{\mathbf{O}} & \to & X_{\mathbf{O}} \\ (g, M) & \mapsto & g \bullet M = \sqrt{g(M^2)} \end{array}$$

Notice that for $k \in F_{4(-52)}$, we have that

$$k \bullet M = \sqrt{k(M^2)} = \sqrt{k(M \star M)} = \sqrt{k(M) \star k(M)} = k(M).$$

PROPOSITION 3.15. *The group $E_{6(-26)}$ acts transitively on $X_{\mathbf{O}}$, and the stabilizer of I is $F_{4(-52)}$.*

PROOF. Since $k \bullet I = k(I) = I$ for $k \in F_{4(-52)}$, the stabilizer of I contains $F_{4(-52)}$, and conversely, for $g \in E_{6(-26)}$, if $g(I) = I$ then $g \in F_{4(-52)}$ by definition. The action is transitive because given any $M \in X_{\mathbf{O}}$, there exists a $k \in F_{4(-52)}$ such that $k \bullet M = D$ is diagonal. Since $\det(M) = \det(k(M)) = 1$, the element D^{-1} defines an element in $E_{6(-26)}$ (coming from $SL_3(\mathbf{R})$), and $D^{-1} \bullet D = I$. □

REMARK 3.16. For any two elements M, N of $X_{\mathbf{O}}$, there exists $g \in E_{6(-26)}$ such that $M = g \bullet I$ and $N = g \bullet D$ where D is a diagonal matrix in $\mathcal{J}$. Indeed, in the proof of the preceding proposition, we saw that there exists an $h \in E_{6(-26)}$ with $I = h \bullet M$, and that any matrix in $\mathcal{J}$ (and thus in particular $h \bullet N$) is diagonalizable by elements of $F_{4(-52)}$. We then choose $k \in F_{4(-52)}$ such that $k(h \bullet N) = D$ is diagonal, and set $g = (kh)^{-1}$.

DEFINITION 3.17. For a diagonal element D, we set

$$\|D\| = \max\{d_i | i = 1, 2, 3\}$$

where the d_i's denote the elements in the diagonal of D. We equip $X_{\mathbf{O}}$ with the distance

$$d(M, N) = \log \|D\| + \log \|D^{-1}\|.$$

for D as in the previous remark. It is well defined because of point 3) of Remark 3.13.

We will now describe some other elements in $F_{4(-52)}$ that will later be useful to embed $SL_3(\mathbf{H})$ in $E_{6(-26)}$ starting from an embedding of $\mathbf{H}$ in $\mathbf{O}$. There are a lot of those, but let us now consider the one mapping 1 to e_0, i to e_1, j to e_2 and k to e_6. One can check that we get a decomposition $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}\ell$ (where $\ell = e_3$), and we will, for this part, stick to that way of looking at $\mathbf{H}$ inside $\mathbf{O}$. Now, given a unit $h \in \mathbf{H}$ we define two maps

$$\begin{array}{rcl} t_h, u_h : \mathbf{O} & \rightarrow & \mathbf{O} \\ x = a + b\ell & \mapsto & t_h(x) = ha + b\ell \\ & & u_h(x) = a + (b\overline{h})\ell \end{array}$$

and an element (that we will later prove to belong to $F_{4(-52)}$)

$$\begin{array}{ccc} T_h : \mathcal{J} & \rightarrow & \mathcal{J} \\ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} & \mapsto & \begin{pmatrix} \xi_1 & t_h(x_3) & \overline{t_h(\overline{x}_2)} \\ \overline{t_h(x_3)} & \xi_2 & \overline{u_h(\overline{x}_1)} \\ t_h(\overline{x}_2) & u_h(\overline{x}_1) & \xi_3 \end{pmatrix} \end{array}$$

We have to check that $\det \bullet T_h = \det$. Since h has been chosen of norm one, the maps t_h and u_h are just rotations of $\mathbf{O}$ (i.e. elements of $SO_8(\mathbf{R})$), and thus isometries of $\mathbf{O}$, so it remains to prove that $\Re(x_1 x_2 x_3) = \Re(\overline{u_h(\overline{x}_1)}\, \overline{t_h(\overline{x}_2)}\, t_h(x_3))$ for all $x_1, x_2, x_3 \in \mathbf{O}$. This will be done by direct computation, using that for $x = a + b\ell$ and $y = c + d\ell$, the product xy reads $(ac - \overline{d}b) + (b\overline{c} + da)\ell$ (see [**27**]). We now compute, for $x, y \in \mathbf{O}$:

$$\begin{aligned}
t_h(x)u_h(y) &= (ha + b\ell)(c + (d\overline{h})\ell) = (hac - \overline{(d\overline{h})}b) + (b\overline{c} + d\overline{h}ha)\ell \\
&= h(ac - \overline{d}b) + (b\overline{c} + da)\ell = t_h(xy) \\
\overline{t_h(\overline{x})} &= \overline{t_h(\overline{a} - \ell\overline{b})} = \overline{t_h(\overline{a} - b\ell)} \\
&= \overline{h\overline{a} - b\ell} = a\overline{h} + \ell\overline{b} = a\overline{h} + b\ell \\
\overline{t_h(\overline{x})}t_h(y) &= (a\overline{h} + b\ell)(hc + d\ell) = (a\overline{h}hc - \overline{d}b) + (b\overline{c}\overline{h} + da\overline{h})\ell \\
&= u_h(xy)
\end{aligned}$$

Which means that $\overline{u_h(\overline{x}_1)}\,\overline{t_h(\overline{x}_2)} = \overline{t_h(\overline{x}_2)u_h(\overline{x}_1)} = \overline{t_h(\overline{x}_2\,\overline{x}_1)} = \overline{t_h(\overline{x_1x_2})}$, and finally:

$$\Re(\overline{u_h(\overline{x}_1)}\,\overline{t_h(\overline{x}_2)}\,t_h(x_3)) = \Re(\overline{t_h(\overline{x_1x_2})}t_h(x_3)) = \Re(u_h(x_1x_2x_3))$$

This proves the sought equality noticing that the map u_h is the identity on the first copy of $\mathbf{H}$, and in particular on e_0.

Now that we know that $T_h \in E_{6(-26)}$, it is pretty clear that $T_h(I) = I$, and thus $T_h \in F_{4(-52)}$.

Remark 3.18. Any embedding of $\mathbf{H}$ in $\mathbf{O}$ gives a decomposition $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}\ell$ (where ℓ is any imaginary unit in the orthogonal complement of the embedded copy of $\mathbf{H}$ in $\mathbf{O}$).

We will now, for any given embedding φ of $\mathbf{H}$ in $\mathbf{O}$, define an explicit embedding $\widetilde{\varphi}$ of $SL_3(\mathbf{H})$ in $E_{6(-26)}$ compatible with the embedding $\overline{\varphi} : X_{\mathbf{H}} \to X_{\mathbf{O}}$ defined by $\overline{\varphi}(m_{ij}) = (\varphi(m_{ij}))$ for $(m_{ij}) \in X_{\mathbf{H}}$.

Lemma 3.19. *Let S denote the subgroup of $E_{6(-26)}$ generated by elements of the type $x \in SL_3(\mathbf{R})$ and T_h for h a unit in $\mathbf{H}$ as just described. Then S is isomorphic to $SL_3(\mathbf{H})$ and $S(X_{\mathbf{H}}) \subset X_{\mathbf{H}}$, i.e. S stabilizes the embedded copy of $X_{\mathbf{H}}$ in $X_{\mathbf{O}}$.*

Proof. We will first see that the map $\widetilde{\varphi} : SL_3(\mathbf{H}) \to S$ defined on generators by $\widetilde{\varphi}(x) = x$ for any $x \in SL_3(\mathbf{R})$ and $\widetilde{\varphi}(\mathcal{T}_h) = T_h$ is a well defined group isomorphism. We recall that for any unit h in $\mathbf{H}$,

$$\mathcal{T}_h = \begin{pmatrix} h & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines an element of $SL_3(\mathbf{H})$, so that the image of $\widetilde{\varphi}$ is contained in S. The map $\widetilde{\varphi}$ being defined on generators, it is a group homomorphism.

Our map $\widetilde{\varphi}$ is clearly injective, and because of Remark 3.6 we know that $\widetilde{\varphi}$ maps the generators of $SL_3(\mathbf{H})$ onto those of S. Finally, it follows from the definition of T_h that $T_h(X_{\mathbf{H}}) \subset X_{\mathbf{H}}$, and since anyway $SL_3(\mathbf{R})(X_{\mathbf{H}}) \subset X_{\mathbf{H}}$ this finishes the proof. □

REMARKS 3.20. The following embedding of $SL_3(\mathbf{H})$ in $E_{6(-26)}$ has been mentioned by G. Prasad:

For any embedding of $\mathbf{H}$ in $\mathbf{O}$, define

$$\begin{array}{rcl}\Psi : SL_1(\mathbf{H}) & \rightarrow & E_{6(-26)} \\ h & \mapsto & \psi_h\end{array}$$

which is an injective homomorphism (where ψ_h are the elements of $F_{4(-52)}$ described under Remark 3.13). Let

$$C = \{g \in E_{6(-26)} | g\psi_h = \psi_h g \text{ for any } h \in SL_1(\mathbf{H})\}$$

be the centralizer of $\Psi(SL_1(\mathbf{H}))$ in $E_{6(-26)}$, then one can show that $C \simeq S \simeq SL_3(\mathbf{H})$.

Before proceeding, let us mention that it was not completely obvious to embed $SL_3(\mathbf{H})$ in $E_{6(-26)}$ because the subgroup R consisting of those elements of $E_{6(-26)}$ stabilizing a copy of $X_{\mathbf{H}}$ in $X_{\mathbf{O}}$ is not isomorphic to $SL_3(\mathbf{H})$. The restriction map $R \rightarrow SL_3(\mathbf{H})$ is surjective since $S \subset R$, but has a kernel. To see that, take a unit $a \in \mathbf{H}$ and define $\psi_a : \mathcal{J} \rightarrow \mathcal{J}$ by

$$\psi_a \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & r_a(x_3) & \overline{r_a(x_2)} \\ \overline{r_a(x_3)} & \xi_2 & r_a(x_1) \\ r_a(x_2) & \overline{r_a(x_1)} & \xi_3 \end{pmatrix}$$

where, for $y \in \mathbf{O}$ written $y = y_1 + y_2\ell$, $r_a(y) = y_1 + (ay_2)\ell$. One can check that $\psi_a \in F_{4(-52)}$ and since obviously $\rho(\psi_a)$ is the identity on $X_{\mathbf{H}}$, we see that the kernel of ρ is at least $SL_1(\mathbf{H})$. However, the subgroup consisting of those elements of $E_{6(-26)}$ stabilizing a copy of $X_{\mathbf{C}}$ in $X_{\mathbf{O}}$ is isomorphic to $SL_3(\mathbf{C})$.

PROPOSITION 3.21. *For any three points in $X_{\mathbf{O}}$ there is a totally geodesic embedding of $X_{\mathbf{C}}$ containing those three points.*

PROOF. Let $x, y, z \in X_{\mathbf{O}}$. Up to taking the image by an element of $E_{6(-26)}$ (which is an isometry), we can assume that $x = I$ and y is a diagonal matrix (use Remark 3.16). We write

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ \overline{z_{12}} & z_{22} & z_{23} \\ \overline{z_{13}} & \overline{z_{23}} & z_{33} \end{pmatrix}$$

with $z_{11}, z_{22}, z_{33} \in \mathbf{R}$ and z_{12}, z_{13} and z_{23} in $\mathbf{O}$. Consider the element $k = \varphi_a$ where $a = \overline{z_{12}}/|z_{12}|$ is a unit in $\mathbf{O}$ then $k(z)$ only has real elements except for $k(z)_{13} = \overline{k(z)_{31}}$ and $k(z)_{23} = \overline{k(z)_{32}}$, and $k(d) = d$ for any diagonal element. Applying Theorem 3.11 to $k(z)_{13}$ and $k(z)_{23}$, we get an embedding of $\mathbf{H}$ in $\mathbf{O}$ which contains $k(z)_{13}$ and $k(z)_{23}$, and thus an embedding $\overline{\varphi} : X_{\mathbf{H}} \rightarrow X_{\mathbf{O}}$ whose image contains x, y and z, and which is isometric by definition of the distance on $X_{\mathbf{O}}$.

To see that the embedding is totally geodesic is basically the same argument as in the proof of Proposition 3.10: Take x, y in the image of $\overline{\varphi}$, so that $x = \overline{\varphi}(x')$ and $y = \overline{\varphi}(y')$ for $x', y' \in X_{\mathbf{H}}$. There is a $g \in SL_3(\mathbf{H})$ so that $x' = g(I)$ and $y' = g(a)$ for $a \in A$. Now, the embedding $\overline{\varphi}$ comes from an embedding $\widetilde{\varphi} : SL_3(\mathbf{H}) \to E_{6(-26)}$ described in Lemma 3.19, so that $x = \overline{\varphi}(g(I)) = \widetilde{\varphi}(g)(I)$ and $y = \overline{\varphi}(g(a)) = \widetilde{\varphi}(g)(a)$. Since A is obviously in the image of $\overline{\varphi}$, so will $\widetilde{\varphi}(g)(A)$ be (that is, the flat $\widetilde{\varphi}(g)(A)$ contains both x and y, and is in the image of the embedding).

Finally, let us show that a geodesic γ between x and y lies in any flat containing them. Without loss of generality we can assume that $x = I$ and that $y \in A$. Take z on γ, combining the first part of this proof with Proposition 3.10, we can assume that $z \in X_{\mathbf{C}}$ and use Lemma 3.7 to conclude that z lies in any flat of $X_{\mathbf{C}}$ containing x and y, and thus γ will also lie in a flat of $X_{\mathbf{O}}$ containing both x and y. We conclude that $\gamma \subset \widetilde{\varphi}(g)(A) \subset \overline{\varphi}(X_{\mathbf{C}})$.

We are now reduced to find an embedding of $X_{\mathbf{C}}$ in $X_{\mathbf{H}}$ containing x, y and z, but we get this one because of Proposition 3.10, and combining it with $\overline{\varphi}$ we get the sought embedding. □

REMARK 3.22. Being a totally geodesic subspace with respect to this Finsler norm is a strong requirement. Indeed, let us look at the bloc-diagonal embedding of $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$ in $G = SL_4(\mathbf{R})$ and endow $X = G/K$ (for $K = SO_4(\mathbf{R})$) with the distance

$$d(x, y) = \log \|x^{-1}y\| + \log \|y^{-1}x\|.$$

where $\| \ \|$ denotes the operator norm on $SL_4(\mathbf{R})$ acting on $\mathbf{R}^4$. Then, even though $SO_2(\mathbf{R}) \times SO_2(\mathbf{R}) \subset K$,

$$Y = (SL_2(\mathbf{R})/SO_2(\mathbf{R})) \times (SL_2(\mathbf{R})/SO_2(\mathbf{R}))$$

with the induced metric of X doesn't lie totally geodesic in X as the following computation shows: We set

$$x = K, y = \begin{pmatrix} e^1 & 0 & 0 & 0 \\ 0 & e^{-1} & 0 & 0 \\ 0 & 0 & e^6 & 0 \\ 0 & 0 & 0 & e^{-6} \end{pmatrix} K, z = \begin{pmatrix} e^1 & 0 & 0 & 0 \\ 0 & e^{-2} & 0 & 0 \\ 0 & 0 & e^4 & 0 \\ 0 & 0 & 0 & e^{-3} \end{pmatrix} K$$

clearly $x, y, z \in X$, $x, y \in Y$, and

$$d(x, z) + d(z, y) = 4 - (-3) + (6 - 4) - (-6 + 3) = 7 + 5 = 12 = d(x, y)$$

so that z is on a geodesic path connecting x to y, but $z \notin Y$.

PROPOSITION 3.23 (V. Lafforgue). *Any uniform net in $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$ satisfy property ($\underline{H}$) (discussed in Chapter 2).*

PROOF. (See also Remark 2.4). The distance defined on $X_{\mathbf{R}}$ is given by $d(x, y) = \alpha_1 + \alpha_2(\log(a))$ if $x^{-1}y = k(a)$ where, for $\mathfrak{a}$ such that

$A = \exp(\mathfrak{a})$

$$\begin{array}{ccc} \alpha_1 : \mathfrak{a}^+ & \to & \mathbf{R} \\ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} & \mapsto & a_1 - a_2 \end{array}$$

and

$$\begin{array}{ccc} \alpha_2 : \mathfrak{a}^+ & \to & \mathbf{R} \\ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} & \mapsto & a_2 - a_3 \end{array}$$

is a basis for the root system for $SL_3(\mathbf{R})$. According to S. Helgason in [**10**] (Chapter X, table VI), the restricted root systems of $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$ admit a basis with two elements (of respective multiplicities 1,2,4 and 8), so since all the distances coincide on A, this means that they fulfill the hypothesis of V. Lafforgue's Lemma 3.4 in [**14**], and thus any uniform net in $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$ satisfy property ($\underline{H}$). □

Relation with triples in $SL_3(\mathbf{C})$

In this section we will see how Proposition 3.10 and 3.21 imply that Lemmas 3.5 and 3.7 of V. Lafforgue's paper [**14**] hold. To see how property (RD) can be deduced we refer to Chapter 4. But before proceeding, here are some notations.

Definition 3.24. Let G denote $SL_3(\mathbf{K})$ (for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) or $E_{6(-26)}$ and K denote $SU_3(\mathbf{K})$ or $F_{4(-52)}$, and $X_{\mathbf{K}} = G/K$. Denote by A the diagonal matrices in $X_{\mathbf{K}}$. We recall that, for any $x, y \in X_{\mathbf{K}}$, there exists $g \in G$ so that $x, y \in g(A)$. For any $t \in \mathbf{R}$, $x, y, z \in X_{\mathbf{K}}$, we say that (x, y) is *in position* $(t, 0)$ if there exists $g \in G$ such that

$$g(x) = I\ ,\ g(y) = e^{-t/3} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, there is a unique geodesic connecting x and y. We say that (x, y, z) is an *equilateral triangle* of oriented size t (where $t \in \mathbf{R}$ can be positive or negative) if there exists $g \in G$ such that

$$g(x) = I\ ,\ g(y) = e^{-t/3} \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,\ g(z) = e^{-2t/3} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, one can check that for the distance we are considering, an equilateral triangle (x, y, z) of oriented size t verifies that $d(x, y) = d(y, z) = d(z, x) = t$, and that there is a unique geodesic between x and y, between y and z as well as between z and x. This in fact characterize equilateral triangles. For $\delta \geq 0$, we will say that a triple (x, y, z) in $X_{\mathbf{K}}$ *δ-retracts* on another triple (x', y', z') if the paths $xx'y'y$, $yy'z'z$, $zz'x'x$ are δ-paths.

LEMMA 3.25. *Let $\mathbf{K} = \mathbf{H}$ or $\mathbf{O}$. For any $\delta_0 > 0$ there exists a $\delta > 0$ such that the following is true:*

1) For any $x, y, z \in X_{\mathbf{K}}$ there exists x', y', z' an equilateral triangle in $X_{\mathbf{K}}$ such that the triple (x, y, z) δ-retracts on (x', y', z').

2) For any $s, t \in \mathbf{R}$ of same sign and $z, v, w, y \in X_{\mathbf{K}}$ such that $d(v, w) < \delta_0$, (v, z) is in position $(t, 0)$ and (w, y) is in position $(s, 0)$, the triangle z, v, y is δ-retractable.

PROOF. 1) Because of Proposition 3.10 and 3.21, there is an isometric copy of $X_{\mathbf{C}}$ in $X_{\mathbf{K}}$ containing x, y, z. Thus, because of Lemma 3.6 in [**14**] there exists x', y', z' an equilateral triangle in $X_{\mathbf{C}}$ such that the paths $xx'y'y$, $yy'z'z$ and $zz'x'x$ are δ-paths. Since the embedding is totally geodesic, the triangle x', y', z' will be equilateral in $X_{\mathbf{K}}$ as well.

2) Without loss of generality, we can assume that $w = I$ and that y is diagonal, so that $z, v \in h(A)$ for an $h \in G$ such that $d(h(I), I) \leq \delta_0$. Because of Proposition 3.10 and 3.21 there is a totally geodesic embedding $\overline{\varphi} : X_{\mathbf{C}} \to X_{\mathbf{K}}$ containing $h(I)$, w and y. But since $\overline{\varphi}$ comes from an embedding $\varphi : SL_3(\mathbf{C}) \to G$ we have that:

$$h(I) = \overline{\varphi}(\overline{h}) = \overline{\varphi}(h'(I)) = \varphi(h')(I),$$

for an $\overline{h} \in X_{\mathbf{C}}$ and $h' \in SL_3(\mathbf{C})$. Since φ is the identity on A, the whole $h(A)$ is contained in the image of $\overline{\varphi}$ in $X_{\mathbf{K}}$ and we thus can see z, v, w and y in $X_{\mathbf{C}}$. The respective positions of (v, z) and (w, y) do not change, again because φ is the identity on A, so using Lemma 3.7 in [**14**], the triangle z, v, y is δ-retractable in $X_{\mathbf{C}}$ and since the embedding is totally geodesic, it will be δ-retractable in $X_{\mathbf{K}}$ as well. $\square$

COROLLARY 3.26. *Any cocompact lattice in either $SL_3(\mathbf{H})$ or $E_{6(-26)}$ has property (RD).*

We refer to Chapter 4 for a proof of this Corollary.

CHAPTER 4

Uniform lattices in products of SL_3's and rank one's

In this chapter we will prove property (RD) for any discrete cocompact subgroup Γ of the isometry group of a finite product of type

$$(\mathcal{K} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n, d = d_1 + \cdots + d_n)$$

where the $\mathcal{X}_i$'s are either Gromov hyperbolic spaces, $\tilde{A}_2$-buildings endowed with the graph metric given by the one-skeleton of the building, or $X_{\mathbf{K}}$'s for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$, as described in the previous chapter (that is, with a Finsler norm instead of the usual Riemannian metric considered on a symmetric space). This will be done by using exactly the same techniques V. Lafforgue used in [**14**]. We will see that property (L) described in Chapter 2 is in some sense a special case of what follows.

In particular, since any cocompact subgroup in

$$\mathrm{Iso}(\mathcal{X}_1) \times \cdots \times \mathrm{Iso}(\mathcal{X}_n)$$

is a cocompact subgroup of $\mathrm{Iso}(\mathcal{K})$, the above given result implies that any cocompact lattice in

$$G = G_1 \times \cdots \times G_n$$

has property (RD), where the G_i's are either rank one Lie groups (real or p-adic), $SL_3(F)$ (for F a non archimedean locally compact field), $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$.

Two properties of triples of points

DEFINITION 4.1. A triple $x, y, z \in \mathcal{K}$ forms an *equilateral triangle* if when we write in coordinates $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$, the triples $x_i, y_i, z_i \in \mathcal{X}_i$ form

- an equilateral triangle in the sense given in Definition 3.24 if $\mathcal{X}_i$ is an $X_{\mathbf{K}}$.
- an equilateral triangle in the sense given in [**24**] if $\mathcal{X}_i$ is an $\widetilde{A}_2$-type building (i.e. $d_i(x_i, y_i) = d_i(y_i, z_i) = d_i(z_i, x_i)$ and the segments $[x_i, y_i], [y_i, z_i], [z_i, x_i]$ are uniquely geodesic for the graph theoretical distance associated to the one skeleton of the building).
- a single point (i.e. $x_i = y_i = z_i$) if $\mathcal{X}_i$ is a hyperbolic space.

Since we endowed $\mathcal{K}$ with the ℓ^1 combinations of the norms on the $\mathcal{X}_i$'s, it is obvious that in particular an equilateral triangle will satisfy $d(x,y) = d(y,z) = d(z,x)$. Concerning the orientation, remember that in Definition 3.24 we had two possible orientations for a triangle, positive or negative. In $\mathcal{K}$ we will have much more possible orientations since the orientation of a triangle will depend on its orientation in each non hyperbolic coordinate (in hyperbolic coordinates, the projection is just a point and thus has only one possible orientation). Suppose that among the $\mathcal{X}_i$'s forming $\mathcal{K}$, $\mathcal{X}_1, \dots, \mathcal{X}_m$ are not Gromov hyperbolic (for $0 \leq m \leq n$), and set

$$\mathcal{I} = \{(a_1, \dots, a_m) | a_i \in \{+, -\}\},$$

we then say that an equilateral triangle x, y, z has oriented size $j \in \mathcal{I}$ if in the non hyperbolic components $\mathcal{X}_i$ the triangle x_i, y_i, z_i has the orientation given by a_i for $i = 1, \dots, m$.

LEMMA 4.2. *For any $\delta_0 > 0$, there exists $d > 0$ such that the following is true:*

1) For any $x, y, z \in \mathcal{K}$ there exists x', y', z' an equilateral triangle in $\mathcal{K}$ such that the paths $xx'y'y$, $yy'z'z$ and $zz'x'x$ are d-paths.

2) For any two equilateral triangles x, y, z and a, b, c in $\mathcal{K}$ having same orientation and such that $d(x,a)$, $d(y,b)$ are both less than δ_0, the triangles x, z, c and y, z, c are d-retractable.

PROOF. 1) For any $i = 1, \dots, n$, there exist $\delta_i \geq 0$ and an equilateral triangle x'_i, y'_i, z'_i in $\mathcal{X}_i$ such that the paths $x_i x'_i y'_i y_i$, $y_i y'_i z'_i z_i$ and $z_i z'_i x'_i x_i$ are δ_i-paths. Indeed, this follows from Lemma 3.6 in [**14**] and Lemma 3.25 in case $\mathcal{X}_i$ is $X_{\mathbf{K}}$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$, from Section 3 in [**24**] in case $\mathcal{X}_i$ is an $\widetilde{A}_2$-type building and because of Remark 2.7 in case $\mathcal{X}_i$ is a hyperbolic space. Now, setting $x' = (x'_1, \dots, x'_n), y' = (y'_1, \dots, y'_n)$ and $z' = (z'_i, \dots, z'_n)$, we see that by construction the triangle x', y', z' is equilateral and that the path $xx'y'y$ is a d-paths for any $d \geq \sum_{i=1}^n \delta_i$ since

$$\begin{aligned} d(x,x') + d(x',y') + d(y',y) &= \sum_{i=1}^n d_i(x_i,x'_i) + d_i(x'_i,y'_i) + d_i(y'_i,y_i) \\ &\leq \sum_{i=1}^n (d_i(x_i,y_i) + \delta_i) \leq d(x,y) + d \end{aligned}$$

and similarly for the paths $yy'z'z$ and $zz'x'x$.

2) By symmetry, we only have to prove the result for the triple x, z, c. We have to find $d \geq 0$ and u in $\mathcal{K}$ such that the paths xuz, zuc and cux are d-paths. For any $i = 1, \dots, n$, the triangles x_i, y_i, z_i and a_i, b_i, c_i are equilateral triangles of same orientation, and thus because of Lemma 3.7 in [**14**] and Lemma 3.25 in case $\mathcal{X}_i$ is $X_{\mathbf{K}}$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$

or **O**, trivially in case $\mathcal{X}_i$ is an $\widetilde{A}_2$-type building or a hyperbolic space, the triangle x_i, z_i, c_i is δ_i-retractable, that is there exists a point u_i so that the paths $x_i u_i z_i$, $z_i u_i c_i$ and $c_i u_i x_i$ are δ_i-paths. Now the point $u = (u_1, \ldots, u_n)$ is the sought points on which the triangle x, z, c retracts, for $d \geq \sum_{i=1}^{n} \delta_i$. Indeed, let us check that for the path xuz:

$$\begin{aligned} d(x,u) + d(u,z) &= \sum_{i=1}^{n} (d_i(x_i,u_i) + d(u_i,z_i)) \\ &\leq \sum_{i=1}^{n} (d_i(x_i,z_i) + \delta_i) \leq d(x,z) + d. \end{aligned}$$

and similarly for the paths zuc and cux. □

We now turn to property $(\underline{H})$ for the vertices of an $\widetilde{A}_2$-type building:

LEMMA 4.3. *Let $\mathcal{X}$ be the vertices of an $\widetilde{A}_2$-type building, that we endow with the graph theoretical distance associated to the one skeleton of the building, then $\mathcal{X}$ has property $(\underline{H})$.*

PROOF. According to [**24**], $\mathcal{X}$ has property (H_0), but we need to show that $\mathcal{X}$ has property (H_δ) for any $\delta \geq 0$. Take $x, y \in \mathcal{X}$, $\delta \geq 0$ and $r \geq 0$, we need to estimate the cardinality of t's at a distance to x less than r and such that xty form a δ-path. Take such a t, we claim that there exists $\tilde{t} \in \mathcal{X}$ such that $x\tilde{t}y$ is a 0-path, and $d(\tilde{t}, t) \leq \delta$ (in other words, t is at a uniformly bounded distance of the convex hull of x and y). Indeed, by Section 3 in [**24**], there exists an equilateral triangle x', t', y' of $\mathcal{X}$ (say of size N) such that $xx't't$, $tt'y'y$ and $xx'y'y$ are 0-paths, which means that

$$\begin{aligned} d(x,t) + d(t,y) &= d(x,x') + d(y',y) + 2N + 2d(t,t') \\ &= d(x,y) + N + 2d(t,t'), \end{aligned}$$

and since xty is a δ-path, $d(x,t) + d(t,y) \leq d(x,y) + \delta$, and thus $N + d(t,t') \leq \delta$. We set $\tilde{t} = x'$, so that

$$d(\tilde{t}, t) = d(x', t') + d(t', t) = N + d(t', t) \leq \delta,$$

which establishes the claim. Now, remember that we defined

$$\Upsilon_{(\delta,r)}(x,y) = \{t \in \mathcal{X} | xty \text{ is a } \delta - \text{path}, \, d(x,t) \leq r\},$$

and that $\mathcal{X}$ has property (H_0) means that there exists a polynomial P such that $\sharp\Upsilon_{(0,r)}(x,y) \leq P(r)$. Since $\widetilde{A}_2$-type building are of uniformly bounded geometry, there exists $N_\delta \in \mathbf{N}$ such that $N_\delta \geq |B(x,\delta|$ for every $x \in \mathcal{X}$. We set $P_\delta = N_\delta P$, then because of the claim $\sharp\Upsilon_{(\delta,r)}(x,y) \leq N_\delta \sharp\Upsilon_{(0,r)}(x,y) \leq P_\delta(r)$, which means that $\mathcal{X}$ has property $(\underline{H})$. □

REMARK 4.4. Property (H_δ) (and thus property $(\underline{H})$) fails to be true if $\delta > 0$ and if $\mathcal{X}$ is endowed with the euclidean metric of the building. This is because if we take x, y and t in $\mathcal{X}$ such that xty is a δ-path and $d(x,t) \leq r$, then it is not true that t is at a uniformly bounded of a (now unique) geodesic between x and y, the distance between t and the convex hull of x and y behaves like the square root of $r + \delta$.

How we establish property (RD)

We start by recalling some definitions that can be found in [**14**].

Definition 4.5. Let (X,d) be a metric space and Γ be a discrete group acting freely and by isometries on X. We say that (X,Γ) satisfy property (K) if there exists $\delta \geq 0$, $k \in \mathbf{N}$ and Γ-invariant subsets $\mathcal{T}_1, \ldots, \mathcal{T}_k$, of X^3 such that:

(K_a) There exists $C \in \mathbf{R}_+$ such that for any $(x,y,z) \in X^3$, there exists $i \in \{1,\ldots,k\}$ and $(\alpha,\beta,\gamma) \in \mathcal{T}_i$ such that

$$\max\{d(\alpha,\beta), d(\beta,\gamma), d(\gamma,\alpha)\} \leq \\ \leq C \min\{d(x,y), d(y,z), d(z,x)\} + \delta$$

and $x\alpha\beta y$, $y\beta\gamma z$ $z\gamma\alpha x$ are δ-paths.

(K_b) For any $i \in \{1,\ldots,k\}$ and $\alpha,\beta,\gamma,\gamma' \in X$, if $(\alpha,\beta,\gamma) \in \mathcal{T}_i$ and $(\alpha,\beta,\gamma') \in \mathcal{T}_i$ then the triangles $\alpha\gamma\gamma'$ and $\beta\gamma\gamma'$ are δ-retractable.

Remark 4.6. If we assume that (X,d) has property (L) as defined in Chapter 2, then it has property (K), taking $k = 1$ and $\mathcal{T}_1 = \{(\alpha,\beta,\gamma) \in X^3 | \alpha = \beta = \gamma\}$. Then (K_a) holds because of property (L) whereas (K_b) is just void.

Theorem 4.7 (V. Lafforgue [**14**]). *Let X be a discrete metric space, and let Γ be a group acting freely and isometrically on X. If the pair (X,Γ) satisfies $(\underline{H})$ and (K), then Γ satisfies property (RD).*

The following lemma is the analogue of a part of Theorem 3.3 in [**14**].

Lemma 4.8. *Let Γ be a discrete cocompact subgroup of the isometry group of $\mathcal{K}$, and $Z \subset \mathcal{K}$ be a Γ-invariant uniform net. Let X be a free Γ-space and $\theta : X \to Z$ be a Γ-equivariant map. Endow X with the distance*

$$\theta^*(d)(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 + d(\theta(x),\theta(y)) & \text{if } x \neq y \end{cases}$$

where d is the induced distance of $\mathcal{K}$ on Z. Then the pair (X,Γ) satisfies property (K).

Proof. Take $\delta \geq 6R_X + d$ (for d as in Lemma 4.2 and R_X as in Definition 2.6). We first have to define the Γ-invariant subsets of X^3. Let us consider $\mathcal{I}$ as explained in Definition 4.1, that is, $\mathcal{I}$ is a set of indices running along the possible orientations of equilateral triangles in $\mathcal{K}$. For any $i \in \mathcal{I}$, we define

$$\mathcal{T}_i' = \{(\alpha,\beta,\gamma) \in Z^3 \mid \text{there exists } (a,b,c) \in \mathcal{K}^3 \text{ equilateral triangle with } d(\alpha,a) \le R_Z, d(\beta,b) \le R_Z, d(\gamma,c) \le R_Z\}$$

In other words, $\mathcal{T}_i'$ is the set of triples of Z which are not too far from an equilateral triangle, and since Γ acts on $\mathcal{K}$ by isometries, the sets $\mathcal{T}_i'$ are Γ-invariant. We then set, for any $i \in \mathcal{I}$:

$$\mathcal{T}_i = \theta^{-1}(\mathcal{T}_i').$$

Since θ is Γ-equivariant, $\mathcal{T}_i$ is Γ-invariant for any $i \in \mathcal{I}$. Let us explain why then (X,Γ) satisfy property (K). Because of the distance defined on X, it is enough to prove (K_a) and (K_b) for Z and the sets $\mathcal{T}_i'$.

<u>(K_a)</u>:

Take three points $x,y,z \in X^3$, we have to show that there exists (α,β,γ) in some $\mathcal{T}_i$ so that the triple (x,y,z) retracts on (α,β,γ). But because of part 1) of Lemma 4.2, we know that there exists $(x',y',z') \in \mathcal{K}^3$, forming an equilateral triangle and so that the triple (x,y,z) retracts on (x',y',z'). Now Z being a uniform net in $\mathcal{K}$, there exists (α,β,γ) in Z with $d(\alpha,x') \le R_Z$, $d(\beta,y') \le R_Z$ and $d(\gamma,z') \le R_Z$, so that the triple (α,β,γ) belongs to $\mathcal{T}_i$ for some $i \in \mathcal{I}$. We compute:

$$\begin{aligned}
&d(x,\alpha)+d(\alpha,\beta)+d(\beta,y)\\
&\le d(x,x')+d(x',\alpha)+d(\alpha,x')+d(x',y')+d(y',\beta)+d(\beta,y')+d(y',y)\\
&\le d(x,x')+d(x',y')+d(y',y)+4R_Z \le d(x,y)+d+4R_Z \le \delta,
\end{aligned}$$

and similarly for the paths $y\beta\gamma z$ and $z\gamma\alpha x$.

<u>(K_b)</u>:

Take $i \in \mathcal{I}$ and four points in X defining two triples in $\mathcal{T}_i$, say (α,β,γ) and (α,β,γ'). By symmetry, we only have to prove the result for the triple (α,γ,γ'), i.e. we have to show that the triangles (α,γ,γ') is δ-retractable. By definition of $\mathcal{T}_i$, we can find two equilateral triangles a,b,c and x,y,z such that

$$d(\alpha,a) \le R_Z, d(\beta,b) \le R_Z, d(\gamma',c) \le R_Z$$

and

$$d(\alpha,x) \le R_Z, d(\beta,y) \le R_Z, d(\gamma,z) \le R_Z$$

so that obviously $d(a,x) \le 2R_Z$ and $d(b,y) \le 2R_Z$ and thus applying part 2) of Lemma 4.2 we have the existence of $d \ge 0$, and two points u and v in $\mathcal{K}$ so that the paths xuz, zuc and cux are d-paths. Again, Z being a uniform net in $\mathcal{K}$, we can find u' and v' in Z at respective distances less than R_Z to u and v. We claim that the paths $\alpha u'\gamma$, $\gamma u'\gamma'$ and $\gamma' u'\alpha$ are δ-paths:

$$\begin{aligned} & d(\alpha, u') + d(u', \gamma) \\ \leq\ & d(\alpha, x) + d(x, u) + d(u, u') + d(u', u) + d(u, z) + d(z, \gamma) \\ \leq\ & 4R_Z + d(x, u) + d(u, z) \leq d(x, z) + d + 4R_Z \\ \leq\ & d(x, \alpha) + d(\alpha, \gamma) + d(\gamma, z) + d + 4R_Z \\ \leq\ & d(\alpha, \gamma) + (d + 6R_Z) \leq d(\alpha, \gamma) + \delta \end{aligned}$$

and similarly for the other paths. □

Now, if under the assumptions of this lemma we furthermore assume that $\sharp\theta^{-1}(z) \leq N$ (i. e. θ has uniformly bounded fibers), then Z is a uniform net in X. By Proposition 3.23, Lemma 2.13 and Lemma 4.3, all factors of $\mathcal{K}$ are so that any uniform net has property $(\underline{H})$, and thus by Lemma 2.14, Z has property $(\underline{H})$. Finally, we apply Lemma 2.10 to deduce that X has property $(\underline{H})$ as well.

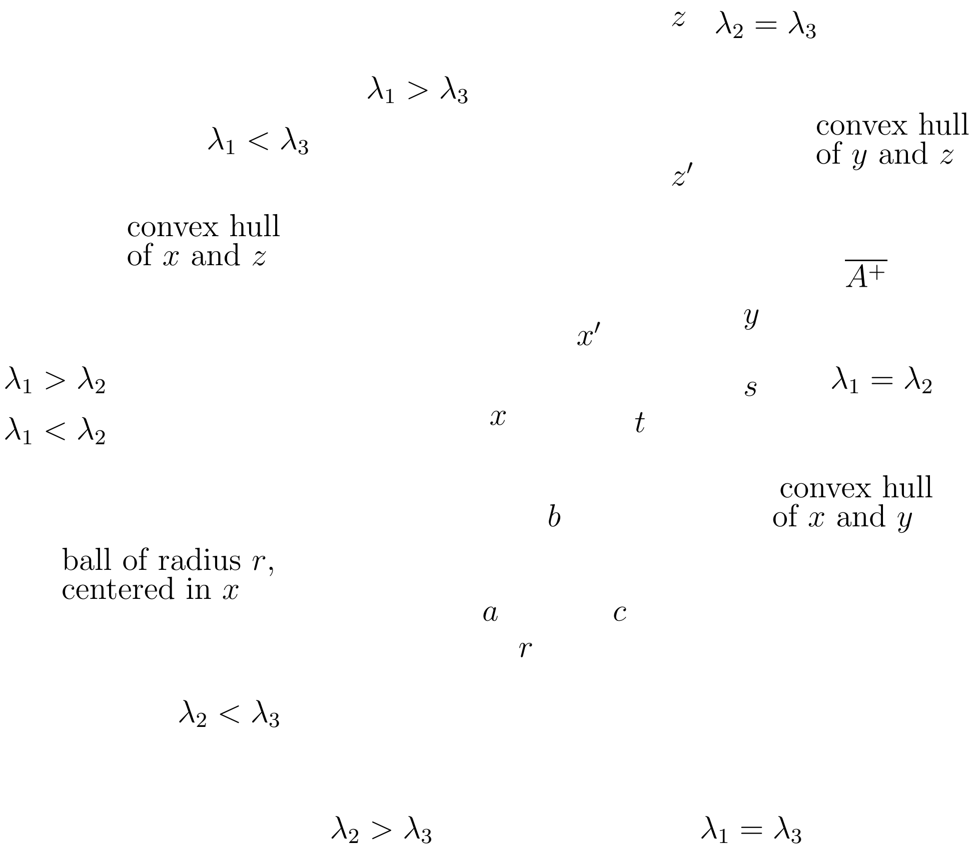
We now can apply Theorem 4.7, as follows: Z is a Γ-invariant uniform net in $\mathcal{K}$ and let $Z = \coprod_{j\in J} \Gamma x_j$ its partition in Γ-orbits. Then with $X = \coprod_{i\in J} \Gamma$ and θ the obvious orbit map and with $\{\mathcal{T}_i\}_{i\in I}$ as defined in the previous lemma we get:

THEOREM 4.9. *Let Γ be a discrete group acting by isometries on $\mathcal{K}$ and with uniformly bounded stabilizers on some Γ-invariant uniform net. Then Γ has property (RD).*

□

If Γ is a cocompact lattice of isometries on $\mathcal{K}$, it is enough to take $Z = \Gamma x_0$, so Γ has property (RD), which finishes the proof of Theorem 0.1.

Flat of type A_2 endowed with a Finsler distance



$$A = \{ \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix} \text{ with } \lambda_1, \lambda_2, \lambda_3 \in \mathbf{R} \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 0\}.$$

$$\overline{A^+} = \{a \in A \text{ such that } \lambda_1 \geq \lambda_2 \geq \lambda_3\}.$$

Assuming that $y = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix} \in A$ and $x = I$, we obtain that $r = d(x, y) = \lambda_1 - \lambda_3 = \lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 = s + t$.

The triple (a, b, c) forms an equilateral triangle.

The triple (x, y, z) retracts on the equilateral triangle (x', y, z').

Loose ends

Un reflet blanc éclaircit les cheveux du voyageur
Mais c'est le givre et non l'âge qui en a changé la couleur.
Il frémit devant sa jeunesse ; la chevelure est demeurée sombre
Long à parcourir est encore le chemin jusqu'à la tombe.

STONE

Quasi-isometry invariance of property (RD). A. Valette in [**2**] asked whether property (RD) is a quasi-isometry invariant. Example 1.18 shows that there is no hope for giving a negative answer to this question using central extensions (as it can be done for property (T)), and hints at a positive answer. Moreover, the only known obstruction to property (RD) being a quasi-isometry invariant, giving a negative answer amounts to finding new obstructions (if any). We point out that it is not known whether property (RD) is a rough-isometry invariant (i. e. invariant by quasi-isometries with multiplicative constant equal one) or even an isometry invariant. Finally notice that properties (H), (K), (L) used in this work are rough-isometries invariants, and this is in my opinion the reason why the whole thing went through products.

Property (RD) for continuous groups down to cocompact subgroups. There is a definition of property (RD) for a continuous locally compact unimodular group G and P. Jolissaint in [**12**] shows that if a discrete cocompact subgroup Γ in G has property (RD) the G will have it as well. The converse is not known (and may not even be true without additional assumptions on G) and would also amount to some kind of rough-isometry invariance since in that case G is roughly isometric to Γ. Notice that V. Lafforgue in [**16**] proved that semisimple Lie groups satisfy property (RD).

Classification of length functions. Is there any way to determine how many equivalence classes of proper length functions (in the sense given under Definition 1.12) on can have on a group Γ and which ones will make how many copies of $\mathbf{Z}$ in G grow exponentially, which ones will be geodesic... For Γ finite, all lengths are obviously equivalent, for $\Gamma = \mathbf{Z}$ we have at least the following equivalence classes: ℓ_0 =word length, $\ell_1 = \log(\ell_0 + 1), \dots, \ell_{n+1} = \log(\ell_n + 1), \dots$ and property (RD) will only hold for ℓ_0 (the other lengths make $\mathbf{Z}$ grow exponentially).

Polynomial cohomology. Let E be a finitely generated abelian group that we endow with its word length ℓ_E, and Γ be a discrete group endowed with a length function ℓ. Denote for $n \in \mathbf{N}$ by $B_\ell^n(r)$ the ball of radius r in Γ^n endowed with the ℓ^1 combination of the length ℓ and set:

$$C^n_{P,\ell}(\Gamma, E) = \{f : \Gamma^n \to E | \exists C, k \in \mathbf{R} : \ell_E\,(f(x)) \leq C(r+1)^k \forall x \in B^n_\ell(r)\}$$

We get a "polynomial cochain" complex and taking coboundaries d in the usual way we can define the polynomial cohomology of (Γ, ℓ) with coefficients in E by

$$H^n_{P,\ell}(\Gamma, E) = \ker(d^n)/\mathrm{im}(d^{n-1})$$

There is a map $H^n_{P,\ell}(\Gamma, E) \to H^n(\Gamma, E)$ coming from the inclusion of the $C^n_{P,\ell}(\Gamma, E)$'s in the usual n-cochains on Γ with coefficients in E. It would be interesting to know whether this map is surjective in degree 2, say for Γ a uniform lattice in $SL_3(\mathbf{R})$. This would imply that any central extension of Γ has property (RD).

Bolicity. In view of Corollary 1.19, V. Lafforgue's Theorem 1.28 and I. Mineyev and G. Yu's Theorem 1.29 it would be interesting to know if there is a way of turning a central extension of a Gromov hyperbolic group into a strongly bolic space.

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Curriculum Vitae

Née à Lausanne le 25 janvier 1973, j'y ai complété ma scolarité obligatoire en 1988 par l'obtention du certificat d'études secondaires, mention scientifique. J'ai obtenu la maturité fédérale de type C (scientifique) en 1991 à Lausanne.

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