Homotopy idempotents on manifolds and Bass' conjectures

A J BERRICK
I CHATTERJI
G MISLIN

The Bass trace conjectures are placed in the setting of homotopy idempotent self-maps of manifolds. For the strong conjecture, this is achieved via a formulation of Geoghegan. The weaker form of the conjecture is reformulated as a comparison of ordinary and L^2–Lefschetz numbers.

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Preface

This note has its origins in talks discussing Bass' trace conjecture. After one such lecture (by IC), R Geoghegan kindly mentioned his geometric perspective on the matter. Then, when another of us (AJB) spoke about the conjecture at the Kinosaki conference, he thought that a topological audience might like to hear about that geometric aspect. Thus, it seemed desirable to attempt to put the conjecture (and its weaker version) in a setting that would be as motivating as possible to topologists. The result of that attempt appears below.

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1 Introduction

In 1976, H Bass [2] conjectured that for any discrete group G, the Hattori–Stallings trace of a finitely generated projective module over the integral group ring of G should be supported on the identity component only. Despite numerous advances (see, for example, Eckmann [6], Emmanouil [10], and our earlier paper [3]), this conjecture remains open in general. In [11], R Geoghegan gave the first topological interpretation, in terms of Nielsen numbers (stated as Theorem 4.4 below). In the setting of selfmaps on manifolds, this translates to the following.

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Theorem 1 *The following are equivalent.*

(a) *The Bass conjecture is a theorem.*

(b) *Every homotopy idempotent selfmap of a closed, smooth and oriented manifold of dimension greater than 2 is homotopic to one that has precisely one fixed point.*

Throughout this paper we use the word "closed" to refer to a connected, compact manifold without boundary. Background material on homotopy idempotents and related invariants will be discussed in Sections 3 and 4. A weaker version of Bass' conjecture amounts to saying that for any group G, the coefficients of the non-identity components of the Hattori–Stallings trace of a finitely generated projective module over the integral group ring of G should sum to zero (and not necessarily be individually zero).

Theorem 2 *The following are equivalent.*

(a) *The weak Bass conjecture is a theorem.*

(b) *Every pointed homotopy idempotent selfmap of a closed, smooth and oriented manifold inducing the identity on the fundamental group, has its Lefschetz number equal to the L^2–Lefschetz number of the induced map on its universal cover.*

Background material regarding L^2–Lefschetz numbers is explained in Section 7. The implication (a) $\Rightarrow$ (b) had already been observed by Eckmann in [7] in a slightly different form. The proofs of these two theorems proceed as follows. Theorem 1 is derived from the analogous statement for finite CW–complexes (involving Nielsen numbers) from Geoghegan's work, which is explained in Section 4. The transition from CW–complexes to manifolds is done in Section 5. The proof of Theorem 1 as well as some applications is discussed in Section 6. For Theorem 2 the strategy is somewhat similar: namely, we first prove the statements for finitely presented groups instead of arbitrary groups and for finite complexes instead of manifolds (see Section 8). To deduce Bass' conjectures (weak and classical) for arbitrary groups we use a remark due to Bass (Lemma 6.3).

2 Review of Bass' conjectures

We briefly recall Bass' conjectures. Let $\mathbb{Z}G$ denote the integral group ring of a group G. The *augmentation trace* is the $\mathbb{Z}$–linear map

$$\epsilon\colon \mathbb{Z}G \to \mathbb{Z}, \quad g \mapsto 1$$

induced by the trivial group homomorphism on G. Writing $[\mathbb{Z}G,\mathbb{Z}G]$ for the additive subgroup of $\mathbb{Z}G$ generated by the elements $gh-hg$ ($g,h\in G$), we identify $\mathbb{Z}G/[\mathbb{Z}G,\mathbb{Z}G]$ (the Hochschild homology group $HH_0(\mathbb{Z}G)$) with $\bigoplus_{[s]\in[G]}\mathbb{Z}\cdot[s]$, where $[G]$ is the set of conjugacy classes $[s]$ of elements s of G. The *Hattori–Stallings trace* of $M=\sum_{g\in G}m_g g\in\mathbb{Z}G$ is then defined by

$$\begin{aligned}\mathrm{HS}(M)&=M+[\mathbb{Z}G,\mathbb{Z}G]\\&=\sum_{[s]\in[G]}\epsilon_s(M)[s]\ \in\ \bigoplus_{[s]\in[G]}\mathbb{Z}\cdot[s],\end{aligned}$$

where for $[s]\in[G]$, $\epsilon_s(M)=\sum_{g\in[s]}m_g$ is a partial augmentation. In particular, the component ϵ_e of the identity element $e\in G$ in the Hattori–Stallings trace is known as the *Kaplansky trace*

$$\kappa\colon\mathbb{Z}G\to\mathbb{Z},\quad \sum m_g g\mapsto m_e.$$

Now, an element of $K_0(\mathbb{Z}G)$ is represented by a difference of finitely generated projective $\mathbb{Z}G$–modules, each of which is determined by an idempotent matrix having entries in $\mathbb{Z}G$. Combining the usual trace map to $\mathbb{Z}G$ of such a matrix with any of the above traces on $\mathbb{Z}G$ turns out to induce a well-defined trace map on $K_0(\mathbb{Z}G)$ that is given the same name and notation as before. Moreover, HS and ϵ are natural with respect to all group homomorphisms (and κ with respect to group monomorphisms). In the case of a free module $\mathbb{Z}G^n$, ϵ takes the value n and so is just the rank of the module.

In [2], Bass conjectured the following.

Conjecture 1 (Classical Bass conjecture) *For any group G, the induced map*

$$\mathrm{HS}\colon K_0(\mathbb{Z}G)\to\bigoplus_{[s]\in[G]}\mathbb{Z}\cdot[s]$$

has image in $\mathbb{Z}\cdot[e]$.

Conjecture 2 (Weak Bass conjecture) *For any group G, the induced maps*

$$\epsilon,\kappa\colon K_0(\mathbb{Z}G)\to\mathbb{Z}$$

coincide.

To clarify the discussion below, it is helpful to consider also reduced K–groups. The inclusion $\langle e\rangle\hookrightarrow G$ induces a natural homomorphism

$$\mathbb{Z}=K_0(\mathbb{Z})=K_0(\mathbb{Z}\langle e\rangle)\longrightarrow K_0(\mathbb{Z}G)$$

whose cokernel is the *reduced* K*–group* $\widetilde{K}_0(\mathbb{Z}G)$, equipped with natural epimorphism $\eta\colon K_0(\mathbb{Z}G) \twoheadrightarrow \widetilde{K}_0(\mathbb{Z}G)$.

3 Homotopy idempotent selfmaps

Let X be a connected CW–complex. A selfmap $f\colon X \to X$ is called *homotopy idempotent* if $f^2 = f \circ f$ is *freely* homotopic to f. Since X is path-connected we can always assume that f fixes a basepoint $x_0 \in X$, so that f induces a (not necessarily idempotent) map $f_\sharp\colon \pi_1(X) \to \pi_1(X)$. Given a homotopy idempotent selfmap $f\colon X \to X$ on a *finite dimensional CW–complex* X, according to Hastings and Heller [14] there is a CW–complex Y and maps $u\colon X \to Y$ and $v\colon Y \to X$ such that the following diagram is (freely) homotopy commutative:

(3.1)
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u"'] \arrow[rrrr, bend left, "f"] & & X \arrow[rr, "f"] \arrow[dr, "u"] & & X \\
& Y \arrow[ur, "v"] \arrow[rr, "\mathrm{id}"] & & Y. \arrow[ur, "v"'] &
\end{tikzcd}$$

In fact, in this diagram we can arrange that the outside triangles strictly commute. By replacing the maps by homotopic ones, we can (and do) choose the maps to preserve basepoints. We then get the following commutative diagram of groups:

$$\begin{tikzcd}
\pi_1(X) \arrow[rr, "f_\sharp"] \arrow[dr, "u_\sharp"'] & & \pi_1(X) \arrow[rr, "f_\sharp"] \arrow[dr, "u_\sharp"] & & \pi_1(X) \\
& \pi_1(Y) \arrow[ur, "v_\sharp"] \arrow[rr] & & \pi_1(Y). \arrow[ur, "v_\sharp"'] &
\end{tikzcd}$$

Here the bottom arrow consists of conjugation by the class of a loop at the basepoint of Y. Looking at the middle triangle, we see that $v_\sharp$ is an injective homomorphism while $u_\sharp$ is surjective; hence we can make the identification

$$\pi_1(Y) \cong v_\sharp(\pi_1(Y)) = \mathrm{Im}(f_\sharp) \le \pi_1(X).$$

If the homotopy idempotent f is a pointed homotopy idempotent (meaning that f^2 is pointed homotopic to f), then $u \circ v\colon Y \to Y$ will induce the identity on $\pi_1(Y)$. If we require that $f_\sharp = \mathrm{id}$, we then get that $\pi_1(Y)$ is isomorphic to $\pi_1(X)$ via $v_\sharp = u_\sharp^{-1}$.

We now explain how, starting from a homotopy idempotent $f\colon X \to X$ of a finite connected complex X with fundamental group $G = \pi_1(X)$, we obtain an element $w(f) \in K_0(\mathbb{Z}G)$. In the situation above, Y is called *finitely dominated*; then the singular chain complex of the universal cover $\widetilde{Y}$ of Y is chain homotopy equivalent to a complex of type FP over $\mathbb{Z}\pi_1(Y)$

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z}$$

with each P_i a finitely generated projective $\mathbb{Z}\pi_1(Y)$–module. We then look at the *Wall element*

$$w(Y) = \sum_{i=0}^{n} (-1)^i [P_i] \in K_0(\mathbb{Z}\pi_1(Y))$$

(where we follow the notation of Mislin [25]). Its image

$$\widetilde{w}(Y) = \eta(w(Y)) \in \widetilde{K}_0(\mathbb{Z}\pi_1(Y))$$

is known as Wall's *finiteness obstruction*, and $\widetilde{w}(Y) = 0$ exactly when Y is homotopy equivalent to a finite complex. Finally, we define

$$w(f) = v_\sharp(w(Y)) \in K_0(\mathbb{Z}G),$$

whose reduction $\widetilde{w}(f) \in \widetilde{K}_0(\mathbb{Z}G)$ was first considered by Geoghegan [11]. As he notes, the element $\widetilde{w}(f)$ "can be interpreted as the obstruction to splitting f through a finite complex". Before proceeding, we check that $w(f)$ is well-defined. First, we observe a form of naturality of Wall elements.

Lemma 3.1 *Let X be a finite n–dimensional complex, and suppose that there are maps (of spaces having the homotopy type of a connected CW–complex)*

$$X \xrightarrow{u} W \xrightarrow{a} V \xrightarrow{v} X$$

such that $a\colon W \to V$ and $u \circ v\colon V \to W$ are homotopy inverse. Then the Wall elements $w(W) \in K_0(\mathbb{Z}\pi_1(W))$ and $w(V) \in K_0(\mathbb{Z}\pi_1(V))$ are related by

$$w(V) = a_\sharp w(W).$$

Proof We use the fact that, because conjugation in G induces the identity map on $K_0(\mathbb{Z}G)$, homotopic maps induce the same homomorphism of K–groups. Recall from Wall [29] that $w(W)$ is defined (uniquely) by means of any n–connected map $\psi\colon L \to W$ where L is a finite n–dimensional complex:

$$w(W) = (-1)^n [\pi_{n+1}(M_\psi, L)]$$

where M_ψ denotes the mapping cylinder of ψ, and the relative homotopy group is considered as a $\pi_1(W)$–module (finitely generated and projective because of the assumption that W is dominated by X). Therefore, to define $w(V)$, we may take

$$w(V) = (-1)^n [\pi_{n+1}(M_{a\psi}, L)].$$

The result then follows from the natural isomorphism of the exact homotopy sequences (of π_1–modules) of the pairs (M_ψ, L) and $(M_{a\psi}, L)$ induced by a. □

We now can see that the obstruction to splitting a homotopy idempotent through a finite complex is well defined.

Lemma 3.2 *Let X be a finite complex with fundamental group G, and for $i = 1, 2$ let $X \xrightarrow{u_i} Y_i \xrightarrow{v_i} X$ be a (homotopy) splitting of a homotopy idempotent map $f\colon X \to X$. Then in $K_0(\mathbb{Z}G)$*

$$v_{1\sharp}(w(Y_1)) = v_{2\sharp}(w(Y_2)).$$

Proof From the homotopy commutative diagram

$$\begin{tikzcd}
 & Y_1 \arrow[rr, "\mathrm{id}"] \arrow[dr, "v_1"] & & Y_1 \arrow[rr, "\mathrm{id}"] \arrow[dr, "v_1"] & & Y_1 \\
X \arrow[ur, "u_1"] \arrow[rr, "f"'] \arrow[dr, "u_2"'] & & X \arrow[ur, "u_1"] \arrow[rr, "f"'] \arrow[dr, "u_2"] & & X \arrow[ur, "u_1"] \arrow[dr, "u_2"] & \\
 & Y_2 \arrow[rr, "\mathrm{id}"'] \arrow[ur, "v_2"] & & Y_2 \arrow[rr, "\mathrm{id}"'] \arrow[ur, "v_2"] & & Y_2
\end{tikzcd}$$

we deduce from a simple diagram chase that $a := u_2 \circ v_1\colon Y_1 \to Y_2$ and $b := u_1 \circ v_2\colon Y_2 \to Y_1$ are mutually inverse homotopy equivalences. Therefore

$$v_2 \circ a \circ u_1\colon X \to Y_1 \to Y_2 \to X$$

is such that a is homotopy inverse to $u_1 \circ v_2\colon Y_2 \to Y_1$; and thus, by Lemma 3.1, $a_\sharp(w(Y_1)) = w(Y_2)$. It then follows that

$$v_{1\sharp}(w(Y_1)) = v_{1\sharp} \circ a_\sharp^{-1}(w(Y_2)) = v_{1\sharp} \circ u_{1\sharp} \circ v_{2\sharp}(w(Y_2)) = f_\sharp \circ v_{2\sharp}(w(Y_2)).$$

Similarly

$$v_{2\sharp}(w(Y_2)) = f_\sharp \circ v_{1\sharp}(w(Y_1)).$$

Then substituting in the previous formula, and using idempotency of $f_\sharp$, gives the result. □

The key fact for giving a topological meaning to the Bass conjecture is the following, which can be extracted from a result of Wall [29, Theorem F] in the light of the above. It is also shown explicitly by Mislin [24].

Theorem 3.3 *Let G be a finitely presented group, let $\widetilde{\alpha} \in \widetilde{K}_0(\mathbb{Z}G)$, and let $n \geq 2$. Then there is a finite n–dimensional complex X^n with fundamental group G and a pointed homotopy idempotent selfmap f of X^n inducing the identity on π_1, such that $\widetilde{w}(f)$ is equal to $\widetilde{\alpha}$.*

Remark 3.4 For $n \geq 3$, the unreduced version of this result also holds. For, given $\alpha \in K_0(\mathbb{Z}G)$, then choose a map f as in the theorem with respect to $\widetilde{\alpha} = \eta(\alpha)$. It follows that for some nonnegative r, s we have $w(f) = \alpha + [\mathbb{Z}G]^r - [\mathbb{Z}G]^s$. Replacing f by $f \vee \mathrm{id}_W$ where $W = \left(\bigvee_r S^3\right) \vee \left(\bigvee_s S^2\right)$ then gives the desired selfmap. When $n = 2$, the method fails, as without the possibility of adjoining a simply-connected space of non-positive Euler characteristic we can only increase the rank of the Wall element. (Recall from Mislin [25, Lemma 5.1] that for any finitely dominated space Y, the rank of $w(Y)$ equals $\chi(Y)$.)

4 Invariants for selfmaps of complexes

We recall from, for example, the articles [11; 12] by Geoghegan, the definition of the *Nielsen number* $N(f)$ of a selfmap $f\colon X \to X$ of a finite connected complex (assumed, as discussed above, to fix a basepoint of X). Let $f_\sharp$ be the endomorphism of $G = \pi_1(X, x)$ induced by f. Define elements α and β of G to be *$f_\sharp$–conjugate* if for some $z \in G$

$$\alpha = z \cdot \beta \cdot (f_\sharp z)^{-1},$$

and let $G_{f_\sharp}$ denote the set of $f_\sharp$–conjugacy classes, making $\mathbb{Z}G_{f_\sharp}$ a quotient of $\mathbb{Z}G$. Now $N(f)$ is defined to be the number of nonzero coefficients in the formula for the *Reidemeister trace* of f at $x \in X$:

$$R(f, x) = \sum\nolimits_{C \in G_{f_\sharp}} n_C \cdot C \in \mathbb{Z}G_{f_\sharp}.$$

The coefficient n_C can be described geometrically as the fixed-point index of a fixed-point class of f, and homologically in terms of traces of the homomorphisms induced by f on the chain complex of the universal cover of X. In the literature, $R(f, x)$ is also known as the *generalized Lefschetz number*.

When, as prompted by Theorem 3.3 above, we take f to induce the identity on π_1, then $R(f, x) \in \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$ and the following holds (cf Geoghegan [12, p505]).

Lemma 4.1 (Geoghegan) *In the setting of diagram (3.1) of Section 3 where X is a finite connected complex, and f is a pointed homotopy idempotent selfmap inducing the identity map on the fundamental group,*

$$\mathrm{HS}(w(f)) = R(f, x).$$

For the computation of Nielsen numbers, the following result also proved in Jiang [17, p20], attributed to Fadell, is useful.

Lemma 4.2 *Suppose that the diagram of finite connected complexes and based maps*

$$\begin{tikzcd} \overline{T} \arrow[r, "\overline{g}"] \arrow[d, "r"] & \overline{T} \arrow[dr, "r"] & \\ T \arrow[r, "g"] & T \arrow[u, "s"] \arrow[r, "\mathrm{id}"] & T \end{tikzcd}$$

is commutative up to (free) homotopy. Then $N(g) = N(\overline{g})$.

Proof We use the definition and notation for $N(g)$ and $N(\overline{g})$ given above; we also put $\overline{G} = \pi_1(\overline{T},\ s(t_0))$ where t_0 is the basepoint of T. Evidently, because $s \circ g \simeq \overline{g} \circ s$, there is a well-defined function

$$s_\sharp \colon G_{g_\sharp} \longrightarrow \overline{G}_{\overline{g}_\sharp}$$

which, because also $r \circ \overline{g} \simeq g \circ r$, while $r \circ s \simeq \mathrm{id}_T$, has left inverse $r_\sharp \colon \overline{G}_{\overline{g}_\sharp} \to G_{g_\sharp}$. In particular, $s_\sharp$ is injective. With reference to the sentence before [11, (2.2)], note that this is not true in general without some condition on s, such as its having a left inverse.

The formula for the Reidemeister trace of g at $t_0 \in T$ is:

$$R(g, t_0) = \sum\nolimits_{C \in G_{g_\sharp}} n_C \cdot C \in \mathbb{Z}G_{g_\sharp}.$$

According to [11, (2.2)], we also have

$$R(\overline{g},\ s(t_0)) = \sum\nolimits_{C \in G_{g_\sharp}} n_C \cdot s_\sharp(C) \in \mathbb{Z}\overline{G}_{\overline{g}_\sharp}.$$

Because $s_\sharp$ is injective, the two sums have the same number of nonzero coefficients; that is, the Nielsen numbers agree. □

The next lemma permits us in our discussion of Nielsen numbers to restrict to the case of those homotopy idempotents that are pointed homotopy idempotents and induce the identity on the fundamental group.

Lemma 4.3 *Suppose that $f \colon X \to X$ is a homotopy idempotent on a finite connected complex X, fixing $x_0 \in X$, with $G = \pi_1(X, x_0)$ and $H := f_\sharp(G)$. Then there is a finite connected complex K with fundamental group isomorphic to H and a pointed homotopy idempotent $g \colon K \to K$, inducing the identity map on H, such that $N(f) = N(g)$. Furthermore, if G satisfies the Bass conjecture, then so does H.*

Proof Let $X \xrightarrow{u} Y \xrightarrow{v} X$ be a splitting for f. Then, by [25, Corollary 5.5], $Y \times S^3$ is homotopy equivalent to a finite connected complex K, because Y is finitely dominated and the Euler characteristic of S^3 is zero. Let $h\colon Y \times S^3 \to K$ be a pointed homotopy equivalence, with pointed homotopy inverse k; define $g\colon K \to K$ to be the map $h \circ (\mathrm{id}_Y \times \{*\}) \circ k$, where $\mathrm{id}_Y \times \{*\}\colon Y \times S^3 \to Y \times S^3$ denotes the idempotent on $Y \times S^3$ given by the projection onto Y. Clearly g is a pointed homotopy idempotent, inducing the identity on the fundamental group of K, and $\pi_1(K) \cong \pi_1(Y) \cong H$.

Writing $u' = u \times \mathrm{id}_{S^3}$ and $v' = v \times \mathrm{id}_{S^3}$, we now apply Lemma 4.2 with $\overline{T} = X \times S^3$ and $T = K$, and maps $\overline{g} = f \times \{*\}$, $r = h \circ u'$, $s = v' \circ k$ and g as defined already. This yields the following homotopy commutative diagram

$$\begin{tikzcd}
X \times S^3 \arrow[r, "f\times\{*\}"] \arrow[d, "u'"] \arrow[dd, bend right, "r"'] & X \times S^3 \arrow[dr, "u'"] \arrow[ddrr, bend left, "r"] & & \\
Y \times S^3 \arrow[d, "h"] & Y \times S^3 \arrow[u, "v'"'] & Y \times S^3 \arrow[dr, "h"] & \\
K \arrow[r, "g"] & K \arrow[u, "k"'] \arrow[uu, bend left, "s"] \arrow[rr, "\mathrm{id}"] & & K.
\end{tikzcd}$$

We conclude that $N(g) = N(f \times \{*\})$.

On the other hand, $N(f \times \{*\}) = N(f)$, as can be seen by again applying Lemma 4.2, with the top part of the diagram as before, but $T = X$, $r = \mathrm{pr}_X$ and s the inclusion $x \mapsto (x, *)$

$$\begin{tikzcd}
X \times S^3 \arrow[r, "f\times\{*\}"] \arrow[d] & X \times S^3 \arrow[dr] & \\
X \arrow[r, "f"] & X \arrow[u] \arrow[r, "\mathrm{id}"] & X.
\end{tikzcd}$$

Therefore $N(f) = N(g)$. That $H \cong \pi_1(Y)$ satisfies the Bass conjecture if G does follows by observing that $v_\sharp\colon \pi_1(Y) \to \pi_1(X)$ is a split injection, and therefore the induced map $HH_0(\mathbb{Z}\pi_1(Y)) \to HH_0(\mathbb{Z}\pi_1(X))$ is a split injection too. $\square$

We are now able to obtain a restatement of the theorem of Geoghegan referred to in the Introduction ([11, Theorem 4.1'] (i)' $\Leftrightarrow$ (iii)'), in a form suitable to the present treatment.

Theorem 4.4 (Geoghegan) *Let G be a finitely presented group. The following are equivalent.*

(a) *G satisfies Bass' Conjecture 1.*

(b) *Every homotopy idempotent selfmap f on a finite connected complex with fundamental group G has Nielsen number either zero or one.*

Proof We start with the implication (a) $\Rightarrow$ (b). Let f be as in (b). Then by Lemma 4.3 we can assume that $f\colon X \to X$ is actually a pointed homotopy idempotent on a finite connected complex X that induces the identity map on $\pi_1(X, x_0) \cong G$. Because G satisfies the Bass conjecture, we have $\mathrm{HS}(w(f)) \in \mathbb{Z}\cdot[e]$. Then, by Lemma 4.1, $R(f, x_0)$ has at most one nonzero coefficient, and $N(f) \leq 1$.

In the other direction, we of course use Theorem 3.3 and Remark 3.4. Then, given $\alpha \in K_0(\mathbb{Z}G)$, there is a finite n–dimensional complex X ($n \geq 3$) with fundamental group G and a pointed homotopy idempotent selfmap f of X inducing the identity on $\pi_1(X) = G$, such that $w(f)$ is equal to α. So, from Lemma 4.1 we deduce that $\mathrm{HS}(\alpha) = R(f, x)$. This last term vanishes when $N(f) = 0$; if $N(f) = 1$, $R(f, x)$ is a nonzero multiple of some class $[s]$, and we are done in case $[s] = [e]$. So, the remaining case is where $R(f, x)$ is a nonzero multiple (necessarily $\chi(Y)$) of some class $[s] \neq [e]$. In that event we may turn instead to $f' = f \vee \mathrm{id}_{S^2}$ with corresponding $Y' = Y \vee S^2$ having $w(f') = w(f) + [\mathbb{Z}G]$. However, this implies the contradiction that $N(f') = 2$, and can therefore be eliminated. □

Remark 4.5 The actual wording of [11, Theorem 4.1'] is in terms of the Bass conjecture for a particular element α of $K_0(\mathbb{Z}G)$, rather than for G itself.

5 Selfmaps of manifolds

From Wecken's work [31], one knows that $N(f)$ serves as a lower bound for the number of fixed points of any map homotopic to f. Thus, the implication (ii) $\Rightarrow$ (i) in the next result is immediate.

Lemma 5.1 *Suppose that $f\colon M \to M$ is a selfmap of a closed manifold M of dimension at least 3. Then the following are equivalent:*

(i) *the Nielsen number of f is 0 or 1;*

(ii) *f is homotopic to a map having one arbitrarily chosen unique fixed point.*

Proof We need only prove that (i) implies (ii).

First, various results in the literature (see in particular Wecken [31], Brown [4], and Shi [27] for the PL case; Jiang [16] for the smooth case) show that every selfmap of

M with Nielsen number N is homotopic to a map with exactly N fixed points. By a result of Schirmer [26, Lemma 2], these fixed points may be chosen arbitrarily.

Second, recall from [26] that every fixed-point-free selfmap of a connected compact PL manifold of dimension at least 3 is homotopic to a selfmap having an arbitrary unique fixed point. The argument there can be adapted as follows.

Choose $a \in M$, and consider the closure $\overline{B}$ of an open ball B around a lying in a chart domain for M. Since f is fixed-point-free, we choose the ball $\overline{B}$ to be small enough so that it is disjoint from its image under f. For convenience, we consider points of $\overline{B}$ with coordinates so that $a = \mathbf{0}$ and $\overline{B}$ consists of those points x with $\|x\| \leq 1$. Now let γ be any path from a to $f(a)$ that continues a unit-speed ray from a to the boundary of $\overline{B}_{1/2}$ (the closed ball of points of B of norm at most $1/2$) and never re-enters $\overline{B}_{1/2}$. Also, let $\lambda\colon M \to [0,1]$ be a function having $\lambda^{-1}(1) = M - B$ and $\lambda^{-1}(0) = \overline{B}_{1/2}$. Then the desired map $g\colon M \to M$ homotopic to f is given as follows.

$$g(x) = \begin{cases} \gamma(t_x) & 0 \leq 2\,\|x\| < 1 \\ f((1-\lambda(x))a + \lambda(x)x) & 1 \leq 2\,\|x\| \leq 2 \\ f(x) & x \in M - \overline{B}\,, \end{cases}$$

where $t_x = 1 - \exp(-2\,\|x\|/(1-2\,\|x\|))$. Here, recall the standard inequality

$$\ln(1-u) > -u/(1-u)$$

for $0 < u < 1$. It implies that, whenever $t_x \neq 0$ and $\gamma(t_x) \in \overline{B}_{1/2}$, so that $\|\gamma(t_x)\| = t_x$,

$$\|\gamma(t_x)\| > 2\,\|x\|\ .$$

Hence a is the unique fixed point of g.

Note that it is possible to make g smooth. For, since every map is homotopic to a smooth map and homotopy does not change Nielsen numbers, there is no loss of generality in assuming f to be smooth. Then, by our taking both γ and λ to be smooth functions in the above argument, a smooth map g results. □

Remark 5.2 Note that this result cannot be extended to dimension 2 in general. Indeed, for every connected, closed surface of negative Euler characteristic and every natural number n, Jiang [18, Theorem 2] exhibits a selfmap f_n of the surface having $N(f_n) = 1$, but with every map homotopic to f_n having more than n fixed points. For some particular results on selfmaps on surfaces, see also Kelly [19].

We next observe that selfmaps of complexes may be studied by means of selfmaps of manifolds without changing the Nielsen number.

Lemma 5.3 *Let X be a finite connected complex. Then the following hold.*

(a) *There is a closed, oriented and smooth manifold M of dimension at least 3 with maps $r\colon M \to X$ and $s\colon X \to M$ having $r \circ s$ pointed homotopic to id_X and inducing isomorphisms of fundamental groups.*

(b) *For any selfmap $f\colon X \to X$, the selfmap $\overline{f} = s \circ f \circ r\colon M \to M$ has Nielsen number*

$$N(\overline{f}) = N(f).$$

(c) *If f is either homotopy idempotent or pointed homotopy idempotent, then so is $\overline{f}$.*

Proof (a) Working up to pointed homotopy type, we may assume that X is a finite simplicial complex of dimension $n \geq 2$. By a result of Wall [30, Theorem 1.4] we can do surgery on the constant map $S^{2n} \to X$ to obtain a smooth, oriented (indeed, stably parallelizable) closed $2n$–manifold M and an n–connected map (called an n–equivalence by Spanier [28]) $r\colon M \to X$. Because $n \geq 2$, the map r is a π_1–isomorphism. Moreover, since the obstruction groups $H^i(Y; \pi_i(r))$ all vanish (or by [28, (7.6.13)]), the identity map $X \to X$ factors up to pointed homotopy through $M \to X$, and the result follows.

(b) This result is immediate from Lemma 4.2 above, on putting $\overline{T} = M$, $T = X$, $g = f$ and $\overline{g} = \overline{f}$.

(c) Obviously,

$$\begin{aligned} \overline{f} \circ \overline{f} &\simeq s \circ f \circ r \circ s \circ f \circ r \\ &\simeq s \circ f \circ f \circ r \simeq s \circ f \circ r \simeq \overline{f}, \end{aligned}$$

and $\overline{f}$ is a pointed idempotent if f is. □

Example 5.4 For any connected non-contractible space X, the monoid of homotopy classes of selfmaps of X always contains at least two idempotents, the class of nullhomotopic maps and the class of maps homotopic to the identity. Each constant map in the former class contains exactly one fixed point (which by connectivity is arbitrary), and obviously has Nielsen number 1.

On the other hand, for X a finite complex the identity map has Nielsen number equal to $\min\{1, |\chi(X)|\}$. When X is also a smooth manifold, it admits a smooth vector field whose only singularity is an arbitrarily chosen point $x_0 \in X$. Its associated flow provides a homotopy from the identity map to a smooth map with sole fixed point x_0.

6 Proof of Theorem 1 and applications

The discussion above now allows a reformulation of statement (b) of Theorem 4.4. Lemma 5.3 combines with Lemma 5.1 to yield a manifold version of statement (b), as follows.

Proposition 6.1 *Let G be a finitely presented group. The following are equivalent.*

(a) *G satisfies Bass' Conjecture 1.*

(b) *Given any closed, smooth and oriented manifold M of dimension at least 3 with $G=\pi_1(M)$, every homotopy idempotent selfmap f on M is homotopic to one that has a single fixed point.*

Remark 6.2 The following facts combine to show that the dimension 3 in (b) above is best possible. For F a closed surface of negative Euler characteristic and $n\geq 2$, Kelly [20] constructs a homotopy idempotent selfmap $f_n\colon F\to F$ such that every map homotopic to f_n has at least n fixed points. On the other hand, the fundamental groups of surfaces are well-known to satisfy Bass' Conjecture 1 (see Eckmann [9], for example).

The following argument of Bass, reported by R Geoghegan, shows that it suffices to consider finitely presented groups in considering Bass' conjectures.

Lemma 6.3 (Bass) *Conjectures 1 and 2 hold for all groups if they hold for all finitely presented groups.*

Proof Fix a group G. We show that any idempotent $\mathbb{Z}G$–matrix A lifts to an idempotent matrix A_1 over the group ring of a finitely presented group G_1. There is a finitely generated subgroup G_0 of G such that the entries of A lie in $\mathbb{Z}G_0$. Write G_0 as F/R where F is a finitely generated free group; and let B be a lift of A to $\mathbb{Z}F$. Then there is a finite subset W of R such that the matrix B^2-B has all its entries in the ideal of $\mathbb{Z}F$ generated by $\{1-r\mid r\in W\}$. Now let $R_1\leq R$ be the normal closure of W in F. Then we have $G_1:=F/R_1$ finitely presented, and the image A_1 of B, with entries in $\mathbb{Z}G_1$, is an idempotent matrix. The map $G_1\twoheadrightarrow G_0\hookrightarrow G$ takes A_1 to A. Therefore $[A_1]\mapsto[A]$ under the induced map $K_0(\mathbb{Z}G_1)\to K_0(\mathbb{Z}G)$. Then the result follows from naturality of HS . □

After Proposition 6.1, it is now straightforward to deduce Theorem 1. □

Remark 6.4 Our arguments lead to variations on Theorem 1. First, one can sharpen the implication (b) $\Rightarrow$ (a) by referring in (b) to a smaller class of manifolds. Because, by Theorem 3.3 above, for a finitely presented group G any $\widetilde{\alpha} \in \widetilde{K}_0(\mathbb{Z}G)$ can be realized by a homotopy idempotent selfmap f of a 2–dimensional complex with fundamental group G (so $\widetilde{w}(f) = \widetilde{\alpha}$), the Bass conjecture is equivalent to the following: *Every homotopy idempotent selfmap of a closed, stably parallelizable smooth* 4*–manifold is homotopic to one with a single fixed point.*

In the other direction, one can strengthen (a) $\Rightarrow$ (b) by enlarging the class of spaces to which (b) applies. There is no need to restrict attention to oriented, smooth manifolds; one can also apply to PL manifolds and other, possibly bounded, Wecken spaces (see Jiang [15]).

As an application of Proposition 6.1 we obtain the following.

Corollary 6.5 *Any homotopy idempotent selfmap on a closed, smooth and oriented* 3*–dimensional manifold* M *is homotopic to one with a single fixed point.*

Proof It is enough to show that the fundamental group G of a closed smooth oriented 3–dimensional manifold M satisfies Bass' conjecture; the Corollary then follows from Proposition 6.1. By Kneser's result (see Milnor [23]), M is a connected sum of prime manifolds M_i, where each M_i belongs to one of the following classes:

(1) M_i with finite fundamental group;

(2) M_i with fundamental group $\mathbb{Z}$;

(3) M_i a $K(\pi, 1)$ manifold (so π is a Poincaré duality group).

Note that the fundamental group of M is the free product of the fundamental groups of the various M_i. By Gersten's result [13], given two groups Γ and H, the reduced projective class group of the free product $\Gamma * H$ reads

$$\widetilde{K}_0(\mathbb{Z}(\Gamma * H)) \cong \widetilde{K}_0(\mathbb{Z}\Gamma) \oplus \widetilde{K}_0(\mathbb{Z}H) .$$

Thus, every element in $\widetilde{K}_0(\mathbb{Z}(\Gamma * H))$ is an integral linear combination of projectives induced up from Γ and H respectively. It follows that if Bass' conjecture holds for both Γ and H, then it holds for $\Gamma * H$ as well.

In the list above, clearly finite groups and $\mathbb{Z}$ satisfy Bass' conjecture. That 3–dimensional Poincaré duality groups satisfy Bass' conjecture follows from Eckmann's work (see Eckmann [9, p247]) on groups of rational cohomological dimension 2. □

Since the Bass conjecture is known for instance for the fundamental groups of manifolds in the class below [9], we have another consequence.

Corollary 6.6 *Any homotopy idempotent selfmap of a non-positively curved, oriented closed manifold of dimension at least* 3 *is homotopic to a map with a single fixed point.*

It would be interesting to see geometric proofs of these facts.

7 Lefschetz numbers

Let X be a CW–complex and $f\colon X \to X$ a continuous selfmap. Then f induces for each $n \in \mathbb{N}$ a map

$$f_n\colon H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})$$

of $\mathbb{Q}$–vector spaces. If the sum of the dimensions of the vector spaces $H_n(X;\mathbb{Q})$ is finite, the *Lefschetz number* of f is defined as

$$L(f) = \sum_{n\geq 0} (-1)^n \mathrm{Tr}(f_n).$$

In cases where the CW–complex X is finite or finitely dominated, the Lefschetz number of a selfmap is obviously always defined, and for $f = \mathrm{id}_X$, $L(f) = \chi(X)$ the Euler characteristic of X. One extends this definition to the case of G–CW–complexes as follows.

Let $\mathcal{N}G$ denote the von Neumann algebra of the discrete group G (*i.e.* the double commutant of $\mathbb{C}G$ considered as a subalgebra of the algebra of bounded operators on the Hilbert space ℓ^2G – see for example Lück [21]). With e as the neutral element of G, write $e \in G \subset \ell^2 G$ for the delta-function

$$e\colon G \to \mathbb{C}, \quad g \mapsto \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

The standard trace

$$\mathrm{tr}_G\colon \mathcal{N}G \to \mathbb{C}, \quad x \mapsto \langle xe, e\rangle_{\ell^2 G} \in \mathbb{C}$$

extends to a trace $\mathrm{tr}_G(\phi) \in \mathbb{C}$ for $\phi\colon M \to M$ a map of finitely presented $\mathcal{N}G$–modules as follows. Recall that a finitely presented $\mathcal{N}G$–module M is of the form $S \oplus T$ with S projective and T of von Neumann dimension 0; the trace of ϕ is then defined as the usual von Neumann trace of the composite $S \to M \to M \to S$ (for the trace of selfmaps of finitely generated projective $\mathcal{N}G$–modules see Lück [21]). It follows that $\mathrm{tr}_G(\mathrm{id}_M) = \dim_G(M)$, the von Neumann dimension of the finitely presented $\mathcal{N}G$–module M (a non-negative real number cf [21]). The Kaplansky trace (as defined in Section 2) induces a trace on G–maps $\psi\colon P\to P$ of finitely generated projective $\mathbb{Z}G$–modules, and

$$\kappa(\psi)=\mathrm{tr}_G(\mathrm{id}_{\mathcal{N}G}\otimes\psi)$$

where $\mathrm{id}_{\mathcal{N}G}\otimes\psi\colon \mathcal{N}G\otimes_{\mathbb{Z}G}P\to\mathcal{N}G\otimes_{\mathbb{Z}G}P$.

Now let Z be a free G–CW–complex that is dominated by a cocompact G–CW–complex (for example, the universal cover of a finitely dominated CW–complex with fundamental group G). A G–map $\tilde f\colon Z\to Z$ induces a map of singular chain complexes $C_*(Z)\to C_*(Z)$ and of L^2–chain complexes

$$C_*^{(2)}(Z):=\mathcal{N}G\otimes_{\mathbb{Z}G}C_*(Z)\to C_*^{(2)}(Z),$$

and therefore of L^2–homology groups

$$H_n^{(2)}(Z):=H_n(Z;\mathcal{N}G)\to H_n^{(2)}(Z).$$

The groups $H_n^{(2)}(Z)$ are finitely presented $\mathcal{N}G$–modules, because the complex $C_*^{(2)}(Z)$ is chain homotopy equivalent to a complex of type FP over $\mathcal{N}G$ and because the category of finitely presented $\mathcal{N}G$–modules is known to be abelian [21]. Thus the induced map

$$\tilde f_n\colon H_n^{(2)}(Z)\to H_n^{(2)}(Z)$$

is a selfmap of a finitely presented $\mathcal{N}G$–module and has, therefore, a well defined trace $\mathrm{tr}_G(\tilde f_n)$ as explained in the beginning of this section; we also write $\beta_n^{(2)}(Z;G)$ for the von Neumann dimension of $H_n^{(2)}(Z)$.

Let Y be a finitely dominated CW–complex with fundamental group G and universal cover Z. Then the n th L^2–Betti number $\beta_n^{(2)}(Y)$ of Y is defined to be $\beta_n^{(2)}(Z;G)$. (If Y happens to be a finite complex, this reduces to the usual L^2–Betti number of Y as defined for instance in Atiyah [1] and Eckmann [8].) By definition, the alternating sum $\sum(-1)^i\beta_i^{(2)}(Y)=\chi^{(2)}(Y)$ is the L^2–Euler characteristic of Y. Recall that for Y a finite complex, $\chi(Y)=\chi^{(2)}(Y)$ by Atiyah's formula [1]; see also Chatterji–Mislin [5] and Lemma 8.1 below for more general results. We now define L^2–Lefschetz numbers as follows.

Definition 7.1 Let Z be be a free G–CW–complex that is dominated by a cocompact G–CW–complex and let $\tilde f\colon Z\to Z$ be a G–map. Denote by $\tilde f_n\colon H_n^{(2)}(Z)\to H_n^{(2)}(Z)$ the induced map in L^2–homology. Then the L^2*–Lefschetz number* of $\tilde f$ is given by

$$L^{(2)}(\tilde f):=\sum_{n\ge 0}(-1)^n\mathrm{tr}_G(\tilde f_n)\in\mathbb{R}.$$

In case Z is cocompact our L^2–Lefschetz number agrees with the one defined by Lück and Rosenberg [22, Remark 1.7]. If Y is a finitely dominated connected CW–complex with fundamental group G, and with universal cover the free G–space $\widetilde{Y}$, then the L^2–Lefschetz number of the identity map of $\widetilde{Y}$ is $\chi^{(2)}(\widetilde{Y};G)=\chi^{(2)}(Y)$, the L^2–Euler characteristic of Y.

8 Proof of Theorem 2

Let Y be a finitely dominated connected CW–complex. Thus $\chi(Y)$ and $\chi^{(2)}(Y)$ are defined as above, and are related as follows.

Lemma 8.1 *Let G be a finitely presented group. Then the following holds.*

(a) *Let Y be a finitely dominated connected CW–complex with fundamental group G. If the finiteness obstruction $\widetilde{w}(Y)\in\widetilde{K}_0(\mathbb{Z}G)$ is a torsion element, then $\chi^{(2)}(Y)=\chi(Y)$.*

(b) *The following are equivalent.*

  (i) *The weak Bass conjecture holds for G.*

  (ii) *For any finitely dominated connected CW–complex Y with $\pi_1(Y)=G$, we have*
$$\chi^{(2)}(Y)=\chi(Y).$$

Proof As in Section 3, for Y finitely dominated, the chain complex $C_*(\widetilde{Y})$ is chain homotopy equivalent to a chain complex P_* of type FP over $\mathbb{Z}G$, $G=\pi_1(Y)$, and we have the Wall element
$$w(Y)=\sum_{i=0}^{n}(-1)^i[P_i]\in K_0(\mathbb{Z}G).$$
As $\mathcal{N}G\otimes_{\mathbb{Z}G}P_*\simeq C^{(2)}(\widetilde{Y})$ and $\mathrm{tr}_G(\mathcal{N}G\otimes_{\mathbb{Z}G}P_i)=\kappa(P_i)$, we see that
$$\chi^{(2)}(Y)=\sum(-1)^i\mathrm{tr}_G(\mathcal{N}G\otimes_{\mathbb{Z}G}P_i)=\sum(-1)^i\kappa(P_i)=\kappa(w(Y)).$$
On the other hand, $\mathbb{Z}\otimes_{\mathbb{Z}G}P_*\simeq C_*(Y)$ and $\epsilon(P_i)=\dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}G}P_i)$ so that
$$\chi(Y)=\sum(-1)^i\dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}G}P_i)=\sum(-1)^i\epsilon(P_i)=\epsilon(w(Y)).$$

(a) Again we observe that for $n>1$,
$$\chi(Y\vee(\vee^k S^n))=\chi(Y)+(-1)^n k$$

and

$$\chi^{(2)}(Y \vee (\vee^k S^n)) = \chi^{(2)}(Y) + (-1)^n k.$$

On the other hand, $w(Y \vee (\vee^k S^n)) = w(Y) + (-1)^n k$, so that without loss of generality we may assume that actually $w(Y)$ is a torsion element. But then, since the range of the Hattori–Stallings trace is torsion-free, Bass' conjectures are valid for torsion elements of $K_0(\mathbb{Z}G)$ and we have

$$\kappa(w(Y)) = \epsilon(w(Y)).$$

Thus,

$$\chi^{(2)}(Y) = \chi(Y),$$

proving the claim.

(b) (i) $\Rightarrow$ (ii): Assuming the weak Bass conjecture, we have

$$\chi^{(2)}(Y) = \kappa(w(Y)) = \epsilon(w(Y)) = \chi(Y).$$

Assuming the Bass conjecture, this implication has also been proved by Eckmann [7].

(ii) $\Rightarrow$ (i): Recall from Theorem 3.3 that for a finitely generated projective $\mathbb{Z}G$–module P there is always a finitely dominated CW–complex Y whose Wall element $w(Y)$ equals $[P] \in K_0(\mathbb{Z}G)$. We then have that

$$\kappa(P) = \kappa(w(Y)) = \chi^{(2)}(Y) = \chi(Y) = \epsilon(w(Y)) = \epsilon(P). \qquad \square$$

Next, we need an intermediate result.

Lemma 8.2 *Let X be a finite connected complex, and $f\colon X \to X$ be a homotopy idempotent. Let Y be a finitely dominated CW–complex determined by f as in Section 3.*

(a) *Then $L(f) = \chi(Y)$.*

(b) *If moreover, f is a pointed homotopy idempotent inducing the identity on $G = \pi_1(X)$ and $\widetilde{f}$ denotes the induced G–map on the universal cover of X, then $L^{(2)}(\widetilde{f}) = \chi^{(2)}(Y)$.*

Proof (a) Applying $H_i(-;\mathbb{Q})$ to the diagram (3.1) of Section 3 yields the following commutative diagram of groups:

$$\begin{tikzcd}
H_i(X) \arrow[rr, "f_i"] \arrow[rrrr, bend left, "f_i"] \arrow[dr, "u_i"'] & & H_i(X) \arrow[rr, "f_i"] \arrow[dr, "u_i"] & & H_i(X) \\
& H_i(Y) \arrow[ur, "v_i"] \arrow[rr, "\mathrm{id}_i"] & & H_i(Y). \arrow[ur, "v_i"'] &
\end{tikzcd}$$

We can now compute that

$$\mathrm{Tr}(f_i) = \mathrm{Tr}(v_i u_i) = \mathrm{Tr}(u_i v_i) = \mathrm{Tr}(\mathrm{id}_i) = \dim(H_i(Y;\mathbb{Q})),$$

so that (a) follows by taking alternating sums.

(b) Here we need to know that f induces the identity map on $\pi_1(X)$, in order to obtain equivariance of the induced maps on the universal covers $\widetilde{X}$ and $\widetilde{Y}$. We apply $H_i^{(2)}$ to diagram (3.1):

$$\begin{tikzcd}
H_i^{(2)}(\widetilde{X}) \arrow[rr, "\widetilde{f}_i"] \arrow[rrrr, bend left, "\widetilde{f}_i"] \arrow[dr, "\widetilde{u}_i"'] & & H_i^{(2)}(\widetilde{X}) \arrow[rr, "\widetilde{f}_i"] \arrow[dr, "\widetilde{u}_i"'] & & H_i^{(2)}(\widetilde{X}) \\
& H_i^{(2)}(\widetilde{Y}) \arrow[ur, "\widetilde{v}_i"'] \arrow[rr, "\mathrm{id}_i"] & & H_i^{(2)}(\widetilde{Y}), \arrow[ur, "\widetilde{v}_i"'] &
\end{tikzcd}$$

and compute

$$\mathrm{tr}_G(\widetilde{f}_i) = \mathrm{tr}_G(\widetilde{v}_i\widetilde{u}_i) = \mathrm{tr}_G(\widetilde{u}_i\widetilde{v}_i) = \mathrm{tr}_G(\mathrm{id}_i) = \dim_G(H_i^{(2)}(\widetilde{Y})),$$

and take alternating sums. The desired equality uses the fact that, given two finitely presented $\mathcal{N}G$–modules A and B, with two maps $\phi\colon A \to B$ and $\psi\colon B \to A$, then $\mathrm{tr}_G(\phi\psi) = \mathrm{tr}_G(\psi\phi)$. □

Proposition 8.3 *Let G be a finitely presented group. The following are equivalent.*

(a) *G satisfies the weak Bass conjecture.*

(b) *Every pointed homotopy idempotent selfmap of a closed, smooth and oriented manifold M with $\pi_1(M) = G$ and inducing the identity on G has its Lefschetz number equal to the L^2–Lefschetz number of the induced G–map on the universal cover of M.*

Proof That (a) implies (b) follows from the implication (a) $\Rightarrow$ (b) in Lemma 8.1, combined with Lemma 8.2. To prove that (b) implies (a), namely that the Lefschetz number information on manifolds is enough to imply the weak Bass conjecture, it suffices to see that for a finite connected complex X of dimension $n \geq 2$ there are a closed smooth oriented manifold M and maps $X \to M$, $M \to X$ inducing isomorphisms of the fundamental groups, and such that $X \to M \to X$ is pointed homotopic to id_X. However, this was already discussed in Lemma 5.3. We then conclude by combining the implication (b) $\Rightarrow$ (a) in Lemma 8.1 with Lemma 8.2. □

Finally, we turn to the proof of Theorem 2. That (a) implies (b) follows from the previous proposition, which also shows that (b) implies (a) for all finitely presented groups, and therefore for all groups via Lemma 6.3. □

Remark 8.4 For each group G, it is evident that the algebraic statement of Conjecture 1 for G implies the statement of Conjecture 2. For our geometric formulations of the conjectures, the implication is less clear. One can approach this problem via work of Lück and Rosenberg on computing L^2–Lefschetz numbers and local degrees [22].

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Department of Mathematics, National University of Singapore
Kent Ridge 117543, Singapore

Department of Mathematics, The Ohio State University
231 W 18th Ave, Columbus OH 43210, USA

Department of Mathematics, ETH Zürich
8092 Zürich, Switzerland

berrick@math.nus.edu.sg, indira@math.ohio-state.edu,
mislin@math.ethz.ch

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