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FLAT BUNDLES WITH COMPLEX ANALYTIC HOLONOMY

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Abstract

Let G be a connected complex Lie group or a connected amenable Lie group. We show that any flat principal G-bundle over any finite CW-complex pulls back to a trivial G-bundle over some finite covering space of the base space if and only if the derived group of the radical of G is simply connected. In particular, if G is a connected compact Lie group or a connected complex reductive Lie group, then any flat principal G-bundle over any finite CW-complex pulls back to a trivial G-bundle over some finite covering space of the base space.

1. Introduction

Let G be a Lie group. A principal G-bundle $P: E \to X$ over a connected CW-complex X admits a *flat structure*, if there is a homomorphism

$$\rho: \pi_1(X) \to G,$$

the *holonomy* of the flat bundle, such that the bundle P is equivalent to the G-bundle $\tilde{X} \times_\rho G \to X$ canonically associated with the universal cover $\tilde{X}$ of X; the notation $\tilde{X} \times_\rho G$ refers to the orbit space of $\tilde{X} \times G$ under the $\pi_1(X)$-action given by

$$\gamma(x, g) = (\gamma x, \rho(\gamma)g).$$

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A *flat* G-bundle is a principal G-bundle, equipped with a flat structure ρ as above. According to [17], a principal G-bundle admits a flat structure if and only if the classifying map $\theta: X \to BG$ factors as

$$X \to B\pi_1(X) \to BG,$$

where the first arrow classifies the universal cover of X and the second one is $B\rho$ for some homomorphism $\rho: \pi_1(X) \to G$. Equivalently, if G^δ denotes the group G with the discrete topology and $\iota: G^\delta \to G$ denotes the identity map, a principal G-bundle over X admits a flat structure, if and only if it is classified by a map $\theta: X \to BG$ which factors through

$$B\iota: BG^\delta \to BG.$$

A principal G-bundle over X is called *virtually trivial* if its pullback to some finite covering space of X is trivial. One can in general not expect a principal G-bundle to be virtually trivial, even if the given bundle admits a flat structure. Examples of flat real vector-bundles over surfaces, which are not virtually trivial as vector-bundles, were given by Milnor [19]. His examples have a non-zero Euler class. Examples of flat real vector-bundles with vanishing Euler class, which are not virtually trivial vector-bundles, were given by Millson [18] and Deligne [4]. On the other hand, a result of Deligne and Sullivan states that any flat complex vector-bundle over any finite CW-complex is virtually trivial as a complex vector-bundle [5]. The following question is thus natural to ask:

Question. Under which conditions on a connected Lie group G is any flat principal G-bundle, over any finite CW-complex, virtually trivial as a principal G-bundle?

Generalizing the proof of the Deligne–Sullivan theorem, we first obtain the following result.

Theorem 1.1 *Let G be a connected complex reductive or a compact Lie group and X a connected finite CW-complex. Let $P: E \to X$ be a flat principal G-bundle. Then, P is virtually trivial as a principal G-bundle.*

Our main result is more general and gives a necessary and sufficient condition on a large class of Lie groups G, for a flat principal G-bundle over a finite complex to be virtually trivial as a G-bundle. Recall that the radical R of a connected Lie group G is its maximal connected normal solvable subgroup. It is always a closed subgroup of G, but its commutator subgroup $[R, R]$ is in general not closed in G.

Theorem 1.2 *Let G be a connected complex Lie group or a connected amenable Lie group. The following conditions are equivalent:*

1. *Any flat principal G-bundle over any finite CW-complex is virtually trivial as a G-bundle.*
2. *The derived subgroup $[R, R]$ of the radical R of G is simply connected.*

The equivalent conditions are satisfied if, moreover, G is linear.

In the solvable case, Theorem 1.2 is due to Goldman in [11]. The finiteness assumption on the CW-complex which is the base of the bundle is not stated explicitly in [11], but is necessary, as the following example shows: the flat principal S^1-bundle over $K(\mathbb{Q}/\mathbb{Z}, 1)$ with classifying map induced by the inclusion $\mathbb{Q}/\mathbb{Z} \subset S^1$ is not a virtually trivial S^1-bundle, because its classifying map

$$K(\mathbb{Q}/\mathbb{Z},\,1) \to BS^1 = K(\mathbb{Z},\,2)$$

corresponds to an element of infinite order in

$$H^2(K(\mathbb{Q}/\mathbb{Z},\,1),\,\mathbb{Z}) \cong \varprojlim \mathbb{Z}/n\mathbb{Z}.$$

Note that for $GL(n,\,\mathbb{C})$ the condition (2) of Theorem 1.2 is true: the radical R of $GL(n,\,\mathbb{C})$ is a complex torus $\mathbb{C}^\times$, thus $[R,\,R]$ is trivial. More generally, if G is complex reductive, its radical is a central complex torus, $R = \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$, thus $[R,\,R] = \{e\}$. An example of a complex analytic amenable group G for which the conditions (1) and (2) fail is $G = H/Z$, where H is the complex Heisenberg group of upper triangular complex 3×3-matrices with 1's on the diagonal, and $Z < H$ an infinite cyclic central subgroup; the radical R of G equals G, and $\pi_1([R,\,R]) = \mathbb{Z}$, as one easily checks.

The implication (1) $\Rightarrow$ (2) holds for *any* connected Lie group. Indeed, (1) implies that the natural map $H^2(BG,\,\mathbb{R}) \to H^2(BG^\delta,\,\mathbb{R})$ is zero and, according to [**3**, Proof of Theorem 2.2], this implies (2); the argument is based on a construction of Goldman [**11**]. Hence, in order to prove Theorem 1.2, it is enough to show that if G is a connected complex Lie group or a connected amenable Lie group and, if the derived subgroup of the radical of G is simply connected, then any flat principal G-bundle over any finite CW-complex is virtually trivial as a G-bundle.

The main steps in the proof of Theorem 1.2 are the following. According to [**3**], all real characteristic classes of a connected Lie group G are bounded, when viewed as classes in $H^*(BG^\delta,\,\mathbb{R})$, if and only if the derived subgroup of the radical of G is simply connected. Combining this fact with Gromov's *Mapping Theorem* [**13**, Section 3.1], we reduce the problem in the complex Lie group case to the case of semisimple groups. A connected complex semisimple or, more generally, reductive Lie group G has a unique complex algebraic structure, and there exists a Chevalley integral group scheme $G_\mathbb{Z}$, whose set of $\mathbb{C}$-points $G_\mathbb{Z}(\mathbb{C})$ in its Lie group topology, $G_\mathbb{Z}(\mathbb{C})_{\mathrm{Lie}}$, is isomorphic to G as a complex Lie group (for the existence of $G_\mathbb{Z}$ see [**6**]; see also [**9**]). As explained in [**8**], this opens the way to the application of Sullivan's completion techniques, in a similar way as in [**5**]: the *Hasse Principle* (Lemma 3.1) allows us to conclude. For the case of an amenable Lie group G, we proceed by mapping G to its complexification G^+, which we show to be a homotopy equivalence (Lemma 2.1). This allows us to reduce the amenable case to the complex Lie group case.

According to Gotô [**12**, Theorem 25], a connected solvable Lie group R is linear if and only if the *closure* of its derived subgroup is simply connected. As the map between fundamental groups

$$\pi_1([R,\,R]) \to \pi_1\left(\overline{[R,\,R]}\right),$$

induced by the inclusion, is one-to-one, it follows that a linear connected complex or amenable Lie group G satisfies the equivalent conditions of the theorem.

We note that there exist *nonlinear* connected amenable Lie groups, which satisfy the equivalent conditions of Theorem 1.2 (the radical R of such a group then satisfies $\pi_1([R,\,R]) = 0$, but $\pi_1(\overline{[R,\,R]}) \neq 0$). The following is an example. Take for H the Heisenberg group of upper triangular real 3×3-matrices with 1's on the diagonal and $Z < H$ an infinite cyclic central subgroup. Embed Z in S^1 and note that the diagonal embedding $Z < S^1 \times H$ identifies Z with a discrete central subgroup, with quotient group $R \coloneqq S^1 \times_Z H$ a nilpotent Lie group, equal to its radical. One checks that $[R, R] \cong \mathbb{R}$ with closure $\overline{[R, R]}$ in R is isomorphic as a Lie group to $S^1 \times S^1$. By Gotô's theorem cited above, we infer that R is not a linear Lie group. It follows that R is a nonlinear connected nilpotent Lie group with derived group $[R, R]$ simply connected.

The paper is organized as follows. In Section 2 we prove the results on the complexification of amenable Lie groups, which we need later on (Lemma 2.1). In Section 3 we prove Theorem 1.1, and in Section 4 we prove a lemma closely related to Goldman's result [11], the main difference being that it also applies to bundles which are not necessarily flat. In Section 5 we prove Theorem 1.2.

2. Complexification of amenable Lie groups

We first recall some general facts on the complexification of a connected Lie group G. We follow the notation used in Hochschild [15] (see also Bourbaki [1 Chapter III, §6, Prop. 20]). To any Lie group corresponds a complex Lie group G^+ and a homomorphism of Lie groups

$$\gamma_G \colon G \to G^+,$$

called the *universal complexification* of G, with the property that, for every continuous homomorphism η of G into a complex Lie group H, there is one and only one complex analytic homomorphism $\eta^+ \colon G^+ \to H$ such that $\eta = \eta^+ \circ \gamma_G$. In general γ_G is not injective. Its kernel is a central (not necessarily discrete) subgroup of G. Let $R < G$ denote the radical of the connected Lie group G and $L < G$ a Levi subgroup (a maximal connected semisimple subgroup). Then $G = RL$ and, in case $L < G$ is closed, the kernel of γ_G coincides with the kernel of γ_L and is discrete in G (see [15, Theorem 4]). Also, if G is linear, γ_G is injective and for G compact, γ_G maps G isomorphically onto a maximal compact subgroup of G^+. Therefore, for G compact, the map γ_G is a homotopy equivalence.

As explained in [15], in the case R is a connected solvable Lie group (not necessarily linear), the complexification map γ_R is injective and induces an isomorphism between fundamental groups $\pi_1(R) \to \pi_1(R^+)$. The universal covers of the solvable Lie groups R and R^+ being contractible, it follows that $\gamma_R \colon R \to R^+$ is a homotopy equivalence.

Lemma 2.1 *Let G be a connected amenable Lie group and $\gamma_G \colon G \to G^+$ the universal complexification map:*

(1) *The radical of G^+ is naturally isomorphic to the complexification R^+ of the radical R of G.*
(2) *$\gamma_G \colon G \to G^+$ is one-to-one and a homotopy equivalence.*
(3) $\pi_1([R, R]) \cong \pi_1([R^+, R^+])$.

Proof. (1) A connected Lie group G is amenable if and only if it fits in a short exact sequence

$$\{1\} \to R \to G \to Q \to \{1\},$$

where R denotes the radical of G and the quotient Q is compact semisimple [21, Corollary 4.1.9]. Let $L < G$ be a Levi subgroup. Since $G/R = Q$ is compact and semisimple, its fundamental group is finite. Thus $L \to Q$, induced by the projection $G \to Q$, is a finite covering space and it follows that L is compact, thus closed in G. Moreover, L is linear and we conclude that $\gamma_L \colon L \to L^+$ is one-to-one. According to [15, Theorem 4, (2) $\Leftrightarrow$ (5)], we conclude that γ_G is injective. Consider the commutative diagram

$$\begin{array}{ccccc} R & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q \\ \simeq \big\downarrow \gamma_R & & \big\downarrow \gamma_G & & \simeq \big\downarrow \gamma_Q \\ R^+ & \xrightarrow{\iota^+} & G^+ & \xrightarrow{\pi^+} & Q^+ . \end{array}$$

As remarked above, γ_R and γ_Q are injective maps and homotopy equivalences. By [15, Theorem 4], the map ι^+ maps R^+ isomorphically onto the radical of G^+, proving (1).

(2) We first claim that G^+/R^+ is isomorphic to Q^+. To see this, we need to verify that this quotient has the universal property of Q^+. Let ν: $G \to G^+ \to G^+/R^+$ be the natural map. Since $R \subset \ker \nu$, we obtain a natural map $\bar{\nu}$: $Q \to G^+/R^+$. Let f: $Q \to C$ be an analytic homomorphism into a complex Lie group C. Then $f \circ \pi : G \to Q \to C$ is trivial on R and extends therefore uniquely to a complex analytic homomorphism $G^+ \to C$, which vanishes on R^+. It follows that the original map f factors uniquely through $\bar{\nu}$: $Q \to G^+/R^+$, showing that $G^+/R^+ \cong Q^+$. Note that both horizontal lines in the diagram above are fibration sequences. We conclude that γ_G must be a homotopy equivalence too, proving (2).

(3) As R is solvable, γ_R is one-to-one, hence so is its restriction η: $[R, R] \to [R^+, R^+]$ to $[R, R]$. The universal property of

$$\gamma_{[R,\, R]}\colon\ [R, R] \to [R, R]^+$$

implies the existence of a complex analytic homomorphism

$$\eta^+\colon\ [R, R]^+ \to [R^+, R^+],$$

such that $\eta^+ \circ \gamma_{[R,\, R]} = \eta$. Taking derivatives at the identities and using the fact that for any real Lie algebra $\mathfrak{r}$, we have

$$[\mathfrak{r} \otimes \mathbb{C}, \mathfrak{r} \otimes \mathbb{C}] = [\mathfrak{r}, \mathfrak{r}] \otimes \mathbb{C},$$

we deduce that η^+ is a local isomorphism, hence a covering homomorphism. This proves (3) in the case R is simply connected. Indeed, the inclusion γ_R: $R \to R^+$ is a homotopy equivalence, hence R^+ is also simply connected, and according to [2, Lemma 6] we have

$$\pi_1([R^+, R^+]) = \pi_1(R^+) \bigcap [R^+, R^+].$$

This shows that $[R^+, R^+]$ is also simply connected, hence η^+ is a global isomorphism. To handle the general case, let us show that the discrete kernel of η^+ is trivial. To that end, we show that the natural embeddings of fundamental groups in the centers of universal covers coincide. Let $\tilde{R}$ be the universal cover of R. It is obvious from the construction of the universal complexification that the universal cover $\tilde{R}^+$ of R^+ coincides with $(\tilde{R})^+$. We have

$$\begin{aligned}\pi_1([R,R]^+) &= \pi_1([R,R]) = \pi_1(R)\cap[\tilde{R},\tilde{R}]\\ &= \pi_1(R)\cap\left[\tilde{R},\tilde{R}\right]^+ = \pi_1(R^+)\cap\left[\tilde{R},\tilde{R}\right]^+\\ &= \pi_1(R^+)\cap\left[\left(\tilde{R}\right)^+,\left(\tilde{R}\right)^+\right] = \pi_1(R^+)\cap[\tilde{R}^+,\tilde{R}^+]\\ &= \pi_1([R^+,R^+]).\end{aligned}$$

The first equality (as well as the fourth one) is true because the embedding of a connected solvable Lie group in its universal complexification is a homotopy equivalence, the second equality (as well as the last one) is a general fact (see [2, Lemma 6]) about closed normal subgroups in Lie groups, the third equality follows from the fact that $[\tilde{R},\tilde{R}]^+\cap\tilde{R}\subset[\tilde{R},\tilde{R}]$, which is deduced from the corresponding inclusion between Lie algebras. The fifth equality is the simply connected solvable case, hence $[\tilde{R},\tilde{R}]^+ = [(\tilde{R})^+,(\tilde{R})^+]$. Thus $\pi_1([R,R])\cong\pi_1([R^+,R^+])$, proving (3). □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be presented in the form of two lemmas (Lemmas 3.2 and 3.3), one dealing with the reductive case and the other one dealing with compact Lie groups.

First, we fix some notation and recall some facts concerning Sullivan's completion functor [20]. Let p be a prime. We will think of Sullivan's p-adic completion as a functor $X\mapsto\hat{X}_p$ on the homotopy category of connected CW-complexes, together with a natural transformation $X\to\hat{X}_p$, which for X a simply connected CW-complex of finite type induces isomorphisms

$$\pi_i\left(X\right)\otimes\hat{\mathbb{Z}}_p\to\pi_i\left(\hat{X}_p\right),\qquad i\geq 2,$$

with $\hat{\mathbb{Z}}_p$ denoting the ring of p-adic integers. We will need the following basic fact from [20, Theorem 3.2].

LEMMA 3.1 (SULLIVAN, *HASSE PRINCIPLE*) *Let X be a finite CW-complex and Y a simply connected CW-complex of finite type. A map*

$$f\colon X\to Y$$

is homotopic to a constant map if and only if for every prime p the map

$$\hat{f}_p\colon X\to Y\to\hat{Y}_p$$

is homotopic to a constant map.

The point here is that the space X in the lemma does not need to be simply connected (or nilpotent).

LEMMA 3.2 *Let G be a connected complex reductive Lie group and X a connected finite CW-complex. Let $P\colon E\to X$ be a flat principal G-bundle. Then P is a virtually trivial G-bundle.*

Proof. We can assume that G is isomorphic to $G_{\mathbb{Z}}(\mathbb{C})_{\mathrm{Lie}}$ for some Chevalley group scheme $G_{\mathbb{Z}}$, where $G_{\mathbb{Z}}(\mathbb{C})_{\mathrm{Lie}}$ stands for the group of $\mathbb{C}$-points of $G_{\mathbb{Z}}$ in its Lie group topology. Let π be the fundamental group of X and $\rho \colon \pi \to G = G_{\mathbb{Z}}(\mathbb{C})_{\text{Lie}}$ the holonomy of the bundle P. Since π is finitely generated, there exists a subring $\Lambda \subset \mathbb{C}$ of finite type over $\mathbb{Z}$ such that the image of ρ is contained in $G_{\mathbb{Z}}(\Lambda)$. We note that such a Λ is a *Jacobson* ring, meaning that every prime ideal of Λ is the intersection of maximal ideals. Indeed, $\mathbb{Z}$ is a Jacobson ring and therefore Λ is too, being a finitely generated $\mathbb{Z}$-algebra (see Eisenbud [7 Chapter I, Thm. 4.19]). It follows that if $\mathbb{Z} \subset \Lambda$ is the natural inclusion and $\mathfrak{m} \subset \Lambda$ a maximal ideal, then $\mathbb{Z} \cap \mathfrak{m} = (s)$ is a maximal ideal in $\mathbb{Z}$ and $\mathbb{F} = \Lambda/\mathfrak{m}$ is a finite extension field of $\mathbb{Z}/(s)$ (cf. [7 Thm. 4.19]). Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ and $H \subset \mathbb{C}$ a strict Henselization of Λ in $\mathbb{C}$, with residue field $\overline{\mathbb{F}}$. We then obtain a diagram of group homomorphisms

$$\begin{array}{ccccccccc} \rho : \pi & \longrightarrow & G_{\mathbb{Z}}(\Lambda) & \longrightarrow & G_{\mathbb{Z}}(H) & \longrightarrow & G_{\mathbb{Z}}(\mathbb{C})_{\text{Lie}} & \xrightarrow{\cong} & G \\ & \searrow^{\phi(s)} & & & \downarrow & & & & \\ & & & & G_{\mathbb{Z}}(\overline{\mathbb{F}}) & & & & \end{array}$$

such that the image of the composite map $\phi(s) \colon \pi \to G_{\mathbb{Z}}(\overline{\mathbb{F}})$ is finite, because π is finitely generated and $G_{\mathbb{Z}}(\overline{\mathbb{F}})$ is a locally finite group. Let $\overline{X}(s)$ be the finite covering space of X corresponding to the kernel of $\phi(s)$. We will show that the bundle P pulled back to $\overline{X}(s)$ is classified by a map $\psi(s) \colon \overline{X}(s) \to BG$ such that

$$\widehat{\psi(s)}_p \colon \overline{X}(s) \to BG \to \widehat{BG}_p$$

is homotopically trivial for all primes p different from s.

For every prime p different from the characteristic s of $\mathbb{F}$ the map $\widehat{\psi(s)}_p \colon \overline{X}(s) \to \widehat{BG}_p$ is homotopically trivial, because up to homotopy it can be factored through the homotopically trivial map $\overline{X}(s) \to BG_{\mathbb{Z}}(H) \to BG_{\mathbb{Z}}(\overline{\mathbb{F}})$, using natural maps

$$\begin{array}{ccccccccc} \overline{X}(s) & \longrightarrow & BG_{\mathbb{Z}}(\Lambda) & \longrightarrow & BG_{\mathbb{Z}}(H) & \longrightarrow & BG_{\mathbb{Z}}(\mathbb{C})_{\text{Lie}} & \longrightarrow & \widehat{BG}_p \\ & & & & \downarrow & & & & \downarrow \simeq \\ & & & & BG_{\mathbb{Z}}(\overline{\mathbb{F}}) & & \longrightarrow & & ((BG_{\overline{\mathbb{F}}})_{\text{et}})\hat{}_p\,. \end{array}$$

Here $(BG_{\overline{\mathbb{F}}})_{\text{et}}$ stands for the etale homotopy type of the simplicial scheme $BG_{\overline{\mathbb{F}}}$ and $((BG_{\overline{\mathbb{F}}})_{\text{et}})\hat{}_p$ denotes its Sullivan p-completion. The homotopy equivalence on the right corresponds to the bottom row of the commutative diagram of Remark (2.5) in [8]. The bottom map is the map described by (2.2) in [8], followed by p-completion.

It remains to deal with the prime $p = s$. By Corollary 14.5 of Eisenbud [7], there is an element $0 \neq a \in \mathbb{Z}$ such that, for every prime t not dividing a, there is a prime ideal $P(t) < \Lambda$ with $P(t) \cap \mathbb{Z} = (t)$. We choose such a prime t different from s and a maximal ideal $\mathfrak{n}$ of Λ containing $P(t)$. Then the finite field $\Lambda/\mathfrak{n}$ has characteristic t different from s. We then obtain as before a finite covering space $\overline{X}(t) \to X$ such that the bundle P pulled back to $\overline{X}(t)$ is classified by a map $\psi(t) \colon \overline{X}(t) \to BG$ satisfying

$$\widehat{\psi(t)}_p \simeq * : \overline{X}(t) \to BG \to \widehat{BG}_p$$

for all primes p different from t. Now choose a common finite covering space $\overline{X}(s, t)$ of $\overline{X}(s)$ and $\overline{X}(t)$. The bundle P pulled back to $\overline{X}(s, t)$ is classified by a map $\psi(s, t)\colon \overline{X}(s, t) \to BG$ such that

$$\widehat{\psi(s, t)}_p \simeq * : \overline{X}(s, t) \to \widehat{BG}_p$$

for all primes p. Because G is connected, BG is simply connected and using that G is homotopy equivalent to a maximal compact subgroup $K < G$, we see that $BG \simeq BK$ is a simply connected space of finite type. We conclude by Lemma 3.1 that the map

$$\psi(s, t)\colon \overline{X}(s, t) \to X \to BG$$

is homotopically trivial, and it follows that the bundle $P\colon E \to X$ is a virtually trivial G-bundle. □

LEMMA 3.3 *Let G be a connected compact Lie group and X a connected finite CW-complex. Let $P\colon E \to X$ be a flat principal G-bundle. Then P is a virtually trivial G-bundle.*

Proof. For G a connected compact Lie group, the natural map $\gamma_G\colon G \to G^+$ from G to its complexification G^+ maps G isomorphically onto a maximal compact subgroup of G^+ (cf. Chapter XVII, Theorem 5.1 of [**14**]). Thus $G^+/\gamma_G(G)$ is a contractible space and, therefore, γ_G and $B\gamma_G$ are homotopy equivalences. Let $\phi_G\colon X \to BG$ be the classifying map for the bundle P. Since G^+ is a connected complex reductive Lie group, we conclude by the previous lemma that there exists a finite covering space $\overline{X} \to X$ such that

$$\overline{X} \to X \to BG \xrightarrow{B\gamma_G} BG^+$$

is homotopic to a constant map. Because $B\gamma_G\colon BG \to BG^+$ is a homotopy equivalence, we conclude that $\overline{X} \to X \to BG$ is homotopically trivial too, proving the assertion of the lemma. □

4. Bundles with solvable holonomy

The following is a characterization of virtually trivial principal bundles over finite connected *CW*-complexes, in case the structural group is a connected solvable Lie group. It can be viewed as a variation of the theorems due to Goldman and Hirsch [**10**, **11**], but without assuming that the bundle in question is flat.

LEMMA 4.1 *Let R be a solvable connected Lie group and $P\colon E \to X$ a principal R-bundle over a connected finite CW-complex X, with $\psi\colon X \to BR$ the classifying map. The bundle P is virtually trivial as an R-bundle if and only if $\psi^*\colon H^2(BR, \mathbb{R}) \to H^2(X, \mathbb{R})$ is the 0-map.*

Proof. If $\psi^* \neq 0$ then for any finite covering space $\pi\colon \overline{X} \to X$ the composition

$$H^2(BR, \mathbb{R}) \xrightarrow{\psi^*} H^2(X, \mathbb{R}) \xrightarrow{\pi^*} H^2(\overline{X}, \mathbb{R})$$

is non-zero too, because $\pi^*\colon H^*(X, \mathbb{R}) \to H^*(\overline{X}, \mathbb{R})$ is injective. It follows that the classifying map $\overline{X} \to BR$ cannot be homotopically trivial, and hence P cannot pull back to a trivial bundle on some finite covering space of X. Conversely, assume that $\psi^* = 0$. Because R is homotopy equivalent to a maximal compact subgroup $T \subset R$, where T is a torus, BR is homotopy equivalent to $K(\mathbb{Z}^n, 2)$ because $\pi_1(T) \cong \mathbb{Z}^n$. It follows that there is a single obstruction $\omega \in H^2(X, \pi_1(R))$ to the existence of a section for P. The kernel of the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is finite (isomorphic to the torsion subgroup of $H_1(X, \mathbb{Z})$), and thus the hypothesis that $\psi^* = 0$ implies that ω must be a torsion class. From the universal coefficient theorem we see that therefore

$$\omega \in \operatorname{Ext}\big(H_1(X, \mathbb{Z}), \pi_1(R)\big) \rightarrowtail H^2(X, \pi_1(R)) \twoheadrightarrow \operatorname{Hom}(H_2(X, \mathbb{Z}), \pi_1(R)).$$

Let $\mathrm{Tor} \subset H_1(X, \mathbb{Z})$ be the finite torsion subgroup and choose a surjection $\theta\colon \pi_1(X) \to \mathrm{Tor}$. Let $f\colon \overline{X} \to X$ denote the covering space corresponding to the kernel of θ. It follows that

$$f^*(\omega) = 0 \in \operatorname{Ext}(H_1(\overline{X}, \mathbb{Z}), \pi_1(R)).$$

But $f^*(\omega)$ is the only obstruction to the existence of a section for the principal R-bundle $f^*P\colon f^*E \to \overline{X}$, showing that f^*P is trivial, and thus completing the proof of the lemma.

5. Proof of Theorem 1.2

We will need the following two auxiliary results.

LEMMA 5.1 *Let R be a solvable connected Lie group and $P\colon E \to Z$ a principal R-bundle over the finite connected CW-complex Z, classified by $\kappa\colon Z \to BR$. Let G be a connected Lie group containing R as a normal, closed subgroup and denote by $\iota\colon R \to G$ the inclusion. Assume that the principal G-bundle over Z classified by $(B\iota) \circ \kappa\colon Z \to BG$ satisfies $\kappa^* \circ (B\iota)^* = 0\colon H^2(BG, \mathbb{R}) \to H^2(Z, \mathbb{R})$. Then, the principal R-bundle P is virtually trivial.*

Proof. Let $Q = G/R$. Since for any connected Lie group the second homotopy group vanishes and the fundamental group is abelian, we have a short exact sequence of abelian groups

$$0 \to \pi_1(R) \to \pi_1(G) \to \pi_1(Q) \to 0,$$

inducing a split short exact sequence of $\mathbb{R}$-vector spaces

$$0 \to \operatorname{Hom}(\pi_1(Q), \mathbb{R}) \to \operatorname{Hom}(\pi_1(G), \mathbb{R}) \xrightarrow{\Phi} \operatorname{Hom}(\pi_1(R), \mathbb{R}) \to 0.$$

For any connected Lie group L, the group $H_2(BL, \mathbb{R})$ is naturally isomorphic to $H_1(L, \mathbb{R}) \cong \pi_1(L) \otimes \mathbb{R}$. It follows that the natural map $(B\iota)^*\colon H^2(BG, \mathbb{R}) \to H^2(BR, \mathbb{R})$ corresponds to the surjective map Φ. Therefore, the vanishing of

$$\kappa^* \circ (B\iota)^*\colon H^2(BG, \mathbb{R}) \to H^2(Z, \mathbb{R})$$

implies the vanishing of

$$\kappa^*\colon H^2(BR, \mathbb{R}) \to H^2(Z, \mathbb{R}).$$

Using Lemma 4.1 we conclude that the principal R-bundle P is virtually trivial. □

LEMMA 5.2 *Let G be a connected Lie group and let R be its radical. Suppose that the derived group $[R, R]$ is simply connected in its Lie group topology and that G/R has a finite fundamental group. Let G^δ denote the group G with the discrete topology. Then, the identity map on the underlying sets $\iota_G\colon G^\delta \to G$ induces the zero map $\iota_G^*\colon H^2(BG, \mathbb{R}) \to H^2(BG^\delta, \mathbb{R})$.*

Proof. There is a short exact sequence of Lie groups

$$R \xrightarrow{\iota} G \xrightarrow{\pi} Q \tag{1}$$

with R the radical of G and Q semisimple.

Split case. We first assume that the short exact sequence (1) is split, with $\sigma\colon Q \to G$ a splitting. For a discrete group D we write $H_b^*(D, \mathbb{R})$ for its bounded real cohomology, and we denote by

$$\theta_D\colon H_b^*(D, \mathbb{R}) \to H^*(D, \mathbb{R})$$

the forgetful map. Because R^δ is an amenable discrete group, the inflation map

$$\pi_b^*\colon H_b^*(Q^\delta, \mathbb{R}) \to H_b^*(G^\delta, \mathbb{R})$$

is an isomorphism (cf. Ivanov [**16**, Theorem 3.8.4], see also Gromov's *Mapping Theorem* [**13**, Section 3.1]). Therefore, the induced maps

$$\pi_b^*\colon H_b^*(Q^\delta, \mathbb{R}) \to H_b^*(G^\delta, \mathbb{R}), \quad \sigma_b^*\colon H_b^*(G^\delta, \mathbb{R}) \to H_b^*(Q^\delta, \mathbb{R})$$

are inverse isomorphisms. We write

$$\pi_\delta^*\colon H^*(BQ^\delta, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}), \quad \sigma_\delta^*\colon H^*(BG^\delta, \mathbb{R}) \to H^*(BQ^\delta, \mathbb{R})$$

and

$$\pi_{\mathrm{top}}^*\colon H^*(BQ, \mathbb{R}) \to H^*(BG, \mathbb{R}), \quad \sigma_{\mathrm{top}}^*\colon H^*(BG, \mathbb{R}) \to H^*(BQ, \mathbb{R})$$

for the maps induced by π and σ, respectively, at the level of cohomology. We then have a commutative diagram

$$\begin{array}{ccc} H^2(BG,\mathbb{R}) & \xleftarrow{\pi_{\mathrm{top}}^*} & H^2(BQ,\mathbb{R}) \\ \downarrow{\scriptstyle \iota_G^*} & & \downarrow{\scriptstyle \iota_Q^*=0} \\ H^2(BG^\delta,\mathbb{R}) & \xleftarrow{\pi_\delta^*} & H^2(BQ^\delta,\mathbb{R}) \\ \uparrow{\scriptstyle \theta_G} & & \uparrow{\scriptstyle \theta_Q} \\ H_b^2(G^\delta,\mathbb{R}) & \underset{\cong}{\xleftarrow{\pi_b^*}} & H_b^2(Q^\delta,\mathbb{R}). \end{array}$$

Notice that $\iota_Q^* = 0$ in the diagram above because, by assumption, $\pi_1(Q)$ is finite, and thus $H^2(BQ, \mathbb{R}) \cong \operatorname{Hom}(\pi_1(Q), \mathbb{R}) = 0$.

Let $x \in H^2(BG, \mathbb{R})$. We need to show that $\iota_G^*(x) = 0$. Since $[R, R]$ is simply connected, $\iota_G^*(x)$ is bounded, meaning that it lies in the image of θ_G (see Theorem 1.1 of [3]). Choose y such that $\theta_G(y) = \iota_G^*(x)$. Because $y = \pi_b^* \sigma_b^*(y)$, we have

$$\iota_G^* x = \theta_G y = \theta_G\left(\pi_b^* \sigma_b^* y\right) = \pi_\delta^* \theta_Q \sigma_b^*\left(y\right) = \pi_\delta^* \sigma_\delta^*\left(\theta_G y\right) = \pi_\delta^*\left(\sigma_\delta^* \iota_G^* x\right) = \pi_\delta^*\left(\iota_Q^* \sigma_{top}^* x\right) = 0,$$

because $\iota_Q^* = 0$.

Non-split case. Now suppose that the exact sequence (1) is non-split. Let $\tilde{Q} \to Q$ be the universal cover. The pullback of $G \to Q$ over $\tilde{Q}$ yields a short exact sequence of Lie groups

$$R \to \bar{G} \to \tilde{Q}$$

which is split because $\tilde{Q}$ is simply connected (see Lemma 14 of [2]). The natural map $p\colon \bar{G} \to G$ is a surjective homomorphism of connected Lie groups with finite kernel K isomorphic to $\pi_1(Q)$. Since BK is $\mathbb{R}$-acyclic, the induced maps

$$p_{\mathrm{top}}^*\colon H^*(BG, \mathbb{R}) \xrightarrow{\cong} H^*(B\bar{G}, \mathbb{R}) \text{ and } p_\delta^*\colon H^*(BG^\delta, \mathbb{R}) \xrightarrow{\cong} H^*(B\bar{G}^\delta, \mathbb{R})$$

are isomorphisms. From the *split case* we infer that $\iota_{\bar{G}}^*\colon H^2(B\bar{G}, \mathbb{R}) \to H^2(B\bar{G}^\delta, \mathbb{R})$ is the zero map, and thus the corresponding map ι_G^* is zero too. □

We are now ready to complete the proof of our main result.

Proof of Theorem 1.2. As explained in the introduction, (1) ⇒ (2) for any connected Lie group. For the proof of (2) ⇒ (1) we first consider the case of a connected complex Lie group G. Its radical R is a complex Lie subgroup and $G/R =: Q$ is complex semisimple, and has therefore a finite fundamental group [14, Chapter XVII, Theorem 2.1]. By assumption, $[R, R]$ is simply connected. Let $p\colon G \to Q$ be the projection and put $\gamma = Bp\colon BG \to BQ$. Then, the map $\gamma \circ \alpha\colon X \to BQ$ classifies a principal Q-bundle over X which admits a flat structure, because P is flat and the diagram

$$\begin{array}{ccc} BG^\delta & \longrightarrow & BQ^\delta \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BQ \end{array}$$

commutes. By Lemma 3.2 we can find a finite connected covering space $\delta\colon Z \to X$ such that the bundle classified by $\gamma \circ \alpha \circ \delta\colon Z \to BQ$ is a trivial Q-bundle. The lifting property of the fibration $BR \to BG \to BQ$ implies that $\alpha \circ \delta\colon Z \to BG$ factors through $Bi\colon BR \to BG$, where $i\colon R \to G$ stands for the inclusion. In other words, there is a map $\kappa\colon Z \to BR$, with $Bi \circ \kappa\colon Z \to BG$ homotopic to $\alpha \circ \delta\colon Z \to BG$. We claim that the (not necessarily flat) principal R-bundle classified by $\kappa\colon Z \to BR$ is virtually trivial. By Lemma 5.1 it suffices to show that

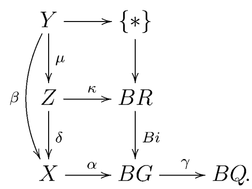
$$(Bi \circ \kappa)^* = (\alpha \circ \delta)^* = 0\colon H^2(BG, \mathbb{R}) \to H^2(Z, \mathbb{R}).$$

As P is flat, $\alpha^*\colon H^2(BG, \mathbb{R}) \to H^2(X, \mathbb{R})$ factors through $H^2(BG^\delta, \mathbb{R})$ and, since by assumption $[R, R]$ is simply connected and Q is complex semisimple, Lemma 5.2 applies and implies that

$$H^2(BG, \mathbb{R}) \to H^2(BG^\delta, \mathbb{R})$$

is the zero map. Thus $(\alpha \circ \delta)* = 0$ and therefore, by Lemma 5.1, the bundle classified by $\kappa\colon Z \to BR$ is virtually trivial. We now choose a finite connected covering space $\mu\colon Y \to Z$ on which the R-bundle pulls back to a trivial bundle, that is, $\kappa \circ \mu \simeq *$. It then follows that the original G-bundle over X pulls back to the trivial bundle over the finite covering space $\beta = \delta \circ \mu\colon Y \to X$.

The following diagram, with commuting squares up to homotopy, depicts, for the convenience of the reader, the maps described above:



This completes the proof for G a connected complex Lie group.

Let us now treat the amenable case. We need to show that for G a connected amenable Lie group with radical R satisfying $\pi_1([R, R]) = 0$, every flat principal G-bundle over a finite CW-complex X is a virtually trivial G-bundle. Let $P\colon E \to X$ be a flat principal G-bundle, which as a G-bundle, is classified by

$$\theta\colon X \to BG.$$

Using the complexification map $\gamma_G\colon G \to G^+$ we obtain an associated flat principal G^+-bundle $P^+\colon E^+ \to X$, classified as a G^+-bundle by the composite map

$$\rho\colon X \xrightarrow{\theta} BG \xrightarrow{B\gamma_G} BG^+.$$

Because of (1) and (3) of Lemma 2.1, we know that G^+ is a complex Lie group with radical isomorphic to R^+ and satisfying $\pi_1([R^+, R^+]) = 0$. Therefore, we infer from the complex case that the bundle classified by ρ is virtually trivial. Because of (2) of Lemma 2.1, we know that $B\gamma_G$ is a homotopy equivalence. It follows that the G-bundle classified by θ is virtually trivial too, finishing the proof of the amenable case.

Finally, it is well known that the radical R of a connected linear Lie group can be embedded as a closed subgroup in a group of upper triangular matrices, which implies that $[R, R]$ is nilpotent and simply connected, finishing the proof of Theorem 1.2. □

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