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# THE FIRST $\ell^2$-BETTI NUMBER AND GROUPS ACTING ON TREES

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*Abstract* We generalize results of Thomas, Allcock, Thom–Petersen, and Kar–Niblo to the first $\ell^2$-Betti number of quotients of certain groups acting on trees by subgroups with free actions on the edge sets of the graphs.

*Keywords:* Cohomology of groups; group actions on trees; presentations of groups

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## 1. Introduction

The $\ell^2$-Betti numbers were introduced by Atiyah as dimensions of heat kernels of certain operators on Riemannian manifolds. The modern formulation assigns $\ell^2$-Betti numbers $b_i^{(2)}(G)$ to arbitrary groups $G$. We refer the reader to Lück's account where the history can be found in the introduction of [12]. Technical results about $\ell^2$-Betti numbers that we need can be found in chapters 6 and 8 of loc. cit. The $\ell^2$-Euler characteristic $\chi^{(2)}(G)$ is defined to be the alternating sum of these Betti numbers when this series is absolutely convergent. Let $\mathfrak{C}$ denote the class of groups $F$ such that

- $\sum_{i\geq 0} b_i^{(2)}(G)$ is finite (this being the condition for absolute convergence),
- $b_1^{(2)}(F) = b_2^{(2)}(F) = 0$, and
- either $\chi^{(2)}(F) = 0$ or $F$ is finite.

Note that $\mathfrak{C}$ contains all $\ell^2$-acyclic groups (i.e. the groups for which $b_i^{(2)} = 0$ for all $i > 0$) and, in particular, it contains all amenable groups. Relevant background on $\ell^2$-cohomology is included in § 2. In this note, we prove the following theorem.

**Theorem 1.1.** *Let $F$ be a group acting simplicially and cocompactly on a simplicial tree, with vertex and edge stabilizers in $\mathfrak{C}$, let $N$ be a subgroup normally generated by $m$ elements, intersecting the vertex stabilizers trivially, and let $G$ denote $F/N$. Then $\chi^{(2)}(F)$ is defined and setting $k := \chi^{(2)}(F) + m$ the following conclusions hold:*

(i) *If $k \leq 0$, then $G$ is infinite.*

(ii) *If $k < 0$, then $b_1^{(2)}(G) \geq -k > 0$.*

(iii) *If $G$ is finite, then $k > 0$ and $|G| \geq \frac{1}{k}$.*

Note that the hypotheses of this theorem guarantee that $N$ acts freely on the specified tree and, in particular, $N$ is necessarily a free group. Note also that, according to [2, Corollary 1.4], if $b_1^{(2)}(G) > 0$ then $G$ has no commensurated infinite amenable subgroup and according to [3, Corollary 6] does not have property (T). If we also have $b_2^{(2)}(G) = 0$, then $G$ is in the class $\mathcal{D}_{\text{reg}}$ by [15, Lemma 2.8]. We refer the reader to [4] for background on property (T) and to [15, Definition 2.6] for the definition of the class $\mathcal{D}_{\text{reg}}$. Acylindrically hyperbolic groups form a large class of groups admitting coarsely proper actions on hyperbolic metric spaces. The class is a generalization of relative hyperbolicity including many Artin groups, mapping class groups, and $\text{Out}(F_n)$. By the main result of Osin's paper [13], we have the following corollary.

**Corollary 1.2.** *Let $G$, $F$ and $N$ be as in Theorem 1.1. Assume that $G$ is finitely presented, (virtually) indicable and that $\chi^{(2)}(F) + m < 0$. Then $G$ is (virtually) acylindrically hyperbolic.*

The simplest way in which the indicability hypothesis may arise is through *stable letters*: Let $T$ denote the $F$-tree of Theorem 1.1. Let $K$ denote the (necessarily normal) subgroup generated by the vertex stabilizers. Then there is a subgroup $E \leq F$ that complements $K$ and all such subgroups are free of uniquely determined rank. Such a subgroup may be referred to as *a subgroup of stable letters of the action*. The group $G$ has an infinite cyclic quotient when $N \cap E$ has infinite index in $E$, in other words, when there is a stable letter that is faithfully represented in $G$, and in this case, $G$ is indicable.

Recall that a group $G$ is $C^*$*-simple* if the reduced group $C^*$-algebra, denoted $C_r^*(G)$, has exactly two norm closed 2-sided ideals, 0, and the algebra $C_r^*(G)$ itself. By [5, Corollary 6.7], we obtain the following.

**Corollary 1.3.** *Let $G$, $F$ and $N$ be as in Theorem 1.1. Then $G$ is $C^*$-simple if and only if it has trivial amenable radical.*

These two corollaries highlight the relation between the first $\ell^2$-Betti number and other areas of geometric group theory. It is an interesting matter for further research to find out whether they lead to new examples.

Theorem 1.1 has some historical pedigree. It originally began life as a result about quotients of free groups due to Thomas (see Theorem 1.4(i)) and was proved using combinatorial methods [16]. The result was generalized by Allcock to incorporate a bound on the rank of the abelianisation of the quotient group [1]. The introduction of $\ell^2$-cohomology came when Peterson–Thom [14, Theorem 3.6] and Kar–Niblo [11] independently linked the inequality of Thomas to the first $\ell^2$-Betti number. These discoveries are summarized in the following result.

**Theorem 1.4 (Thomas [16], Allcock [1], Peterson–Thom [14], Kar–Niblo [11]).** *Let $G$ be a group with a presentation*

$$\langle x_1, \dots, x_n | \, r_1^{k_1}, \dots, r_m^{k_m} \rangle$$

*in which the elements $r_i$ have order $k_i$ when interpreted in $G$.*

(i) *If $n - \sum_{i=1}^m \frac{1}{k_i} \geq 1$ then $G$ is infinite.*

(ii) *If $G$ is finite then $|G| \geq \frac{1}{1-n+\sum_{i=1}^m k_i}$.*

(iii) *If $n - \sum_{i=1}^m \frac{1}{k_i} > 1$ then $G$ is non-amenable.*

**Deduction of Theorem 1.4 from Theorem 1.1.** Let $G$ be a group with a presentation

$$G = \langle x_1, \dots, x_n | \, r_1^{k_1}, \dots, r_m^{k_m} \rangle.$$

Adding $m$ fresh generators $y_1, \dots, y_m$, we can give the following alternative presentation of the same group:

$$G \cong \langle x_1, \dots, x_n, y_1, \dots, y_m | \, y_1^{k_1}, \dots, y_m^{k_m}, r_1 y_1^{-1}, \dots, r_m y_m^{-1} \rangle.$$

Let $F$ be the group with presentation

$$F = \langle x_1, \dots, x_n, y_1, \dots, y_m | \, y_1^{k_1}, \dots, y_m^{k_m} \rangle$$

and let $N$ be the subgroup of $F$ normally generated by $r_1 y_1^{-1}, \dots, r_m y_m^{-1}$. Then $F$ is a free product of cyclic groups which has a natural action on a simplicial tree $T$ (the Bass–Serre tree). In particular, it is virtually free and has Euler characteristic $\chi(F) = \sum_{i=1}^m \frac{1}{k_i} - n - m + 1$ (as explained in [6, Chapter IX.7]). This is equal to $\chi^{(2)}(F)$ by [12, Remark 6.81]. The condition that the $r_i$ have order $k_i$ in the original presentation ensures that $N$ does not meet any of the finite subgroups of $F$, so is torsion-free and acts freely on $T$. Applying Theorem 1.1 with these choices of $F$, $N$, $G$ and $T$ yields Theorem 1.4. □

Throughout this paper, for a group or subgroup $G$, we will adopt the convention that $\frac{1}{|G|}$ be interpreted as zero if $G$ is infinite.

Finally, we compute the first $\ell^2$-Betti number for certain groups acting on trees leading to Theorem 1.5 below, which is proved in § 4. This generalizes a result of Lück [7], which covers the case of an amalgamated free product, and a result of Tsouvalas [17, Corollary 3.7]. Tsouvalas assumes the vertex stabilisers are either residually finite or virtually torsion-free and the edge stabilisers are finite. Here we replace both of these assumptions with Lück's less restrictive assumption that the first $\ell^2$-Betti numbers of the edge stabilisers vanish. So, for example, the theorem applies to fundamental groups of graphs of $\mathfrak{C}$-groups.

**Theorem 1.5.** *Let $F$ be a group acting simplicially on a simplicial tree and let $V$ and $E$ denote sets of representatives of $F$-orbits of vertices and edges. Assume for each $e \in E$ that $b_1^{(2)}(F_e) = 0$, then we have*

$$b_1^{(2)}(F) = \sum_{v \in V} \left( b_1^{(2)}(F_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|} + \frac{1}{|F|}.$$

## 2. Background on $\ell^2$-homology

Let $G$ be a group. Then both $G$ and the complex group algebra $\mathbb{C}G$ act by left multiplication on the Hilbert space $\ell^2 G$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}G$ is the ring of $G$-equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}G$ can be identified with the *ring of affiliated operators*, and is denoted by $\mathcal{U}G$. One has the inclusions $\mathbb{C}G \subseteq \mathcal{N}G \subseteq \ell^2 G \subseteq \mathcal{U}G$ and it is also known that $\mathcal{U}G$ is a self-injective ring which is flat over $\mathcal{N}G$. For more details concerning these constructions, we refer the reader to [12] and especially to Theorem 8.22 of § 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [12, § 8.3]. Now let $Y$ be a $G$-CW complex as defined in [12, Definition 1.25 of § 1.2]. The $\ell^2$-homology groups of $Y$ are then defined to be the equivariant homology groups $H_i^G(Y;\mathcal{U}G)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^G(Y;\mathcal{U}G).$$

The $\ell^2$-Betti numbers of a group $G$ are then defined to be the $\ell^2$-Betti numbers of $EG$, that is to say

$$b_i^{(2)}(G) := b_i^{(2)}(EG). \tag{1}$$

By [12, Theorem 6.54(8)], the zeroeth $\ell^2$-Betti number of $G$ is equal to $1/|G|$. Moreover, if $G$ is finite then $b_n^{(2)}(G) = 0$ for $n \geq 1$.

Let $C_*(Y;\mathcal{U}G)$ denote the standard cellular chain complex of $Y$ with coefficients in $\mathcal{U}G$. We have the formula

$$\dim_{\mathcal{U}G} C_i(Y;\mathcal{U}G) = \sum_{\sigma} \frac{1}{|G_\sigma|}$$

where $\sigma$ runs through a set of orbit representatives of $i$-dimensional cells in $Y$. Suppose that the $\ell^2$-Euler characteristic of $Y$ is defined. Standard arguments of homological algebra give the connection between two Euler characteristic computations (for the details see [12, Lemma 6.80(1)]):

$$\chi^{(2)}(Y) = \sum_i (-1)^i b_i^{(2)}(Y) = \sum_i (-1)^i \dim_{\mathcal{U}G} C_i(Y;\mathcal{U}G) = \sum_i (-1)^i \sum_{\sigma} \frac{1}{|G_\sigma|}. \tag{2}$$

We will need the following lemma for the proofs in the next section. One should think of it as a mild generalization of Theorem 6.54(2) in [12].

**Lemma 2.1 (Comparison with the Borel construction up to rank).** *Let $X$ be a $G$-CW complex. Suppose for all $x \in X$ the isotropy group $G_x$ is finite or $b_p^{(2)}(G_x) = 0$ for all $0 \le p \le n$, then*

$$b_p^{(2)}(X) = b_p^{(2)}(EG \times X) \quad \text{for } 0 \le p \le n.$$

**Proof.** It suffices to prove that the von Neumann dimensions of the kernel and cokernel of the map

$$\mathrm{pr}_p : H_p^G(EG \times X; \mathcal{U}G) \to H_p^G(X; \mathcal{U}G)$$

induced by the projection $EG \times X \to X$ are zero for $0 \le p \le n$. Here $EG \times X$ carries the diagonal action of $G$. By an identical argument to [12, Theorem 6.54(2)], it suffices to prove for each isotropy subgroup $H \le G$ and $0 \le p \le n$ the kernel and cokernel of the map $\mathrm{pr}_p : H_p^H(EH; \mathcal{U}H) \to H_p^H(*; \mathcal{U}H)$ have dimension equal to zero. If $H$ is finite this follows from [12, Theorem 6.54(8a)], and is immediate if $b_p^{(2)}(H) = 0$ for all $0 \le p \le n$. □

## 3. The main theorem

To prove Theorem 1.1, one needs the following method of computing the $\ell^2$-Euler characteristic of a group acting on a tree analogous to Chiswell's result [9] for rational Euler characteristic.

**Proposition 3.1 (Chatterji–Mislin [8]).** *Let $F$ be a group acting on a tree and let $V$ and $E$ denote sets of representatives of $F$-orbits of vertices and edges. If the $\ell^2$-Euler characteristic of each vertex and edge group is finite, then*

$$\chi^{(2)}(F) = \sum_{v \in V} \chi^{(2)}(F_v) - \sum_{e \in E} \chi^{(2)}(F_e).$$

**Proof of Theorem 1.1.** There is a cocompact action of $F$ on a tree $T$ with vertex and edge stabilisers in the class $\mathfrak{C}$. Let $V$ and $E$ denote the vertex and edge sets. Let $\overline{T}$ denote the quotient graph $T/N$ and write $\overline{V}$ and $\overline{E}$ for its vertex and edge sets. Now $G = F/N$ acts cocompactly on $\overline{T}$ with vertex and edge stabilizers in $\mathfrak{C}$. The augmented chain complex of $T$ is the short exact sequence

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$$

of $\mathbb{Z}F$-modules. Restricting to the action of $N$ this short exact sequence leads to a long exact sequence for the homology of $N$. It is straightforward to identify $H_0(N; \mathbb{Z}V)$ with $\mathbb{Z}\overline{V}$ and $H_0(N; \mathbb{Z}E)$ with $\mathbb{Z}\overline{E}$, so that the tail end of the sequence takes the form

$$H_1(N; \mathbb{Z}) \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0. \tag{3}$$

Let $\{r_i : i = 1, ..m\}$ denote a normal generating set for $N$. Choose a vertex $v_0$ in $T$ to be a fixed basepoint. For $1 \le i \le m$, consider the geodesic from $v_0$ to $v_0 r_i$. In the quotient graph $\overline{T}$, this geodesic descends to a loop because $v_0$ and $v_0 r_i$ become identified in $\overline{T}$. Now 2-discs can be glued to each loop. By adjoining free $G$-orbits of 2-discs equivariantly, we can build a 2-complex $Y$ with an action of $G$, whose 1-skeleton is $\overline{T}$, and which has augmented cellular chain complex

$$\mathbb{Z}G^m \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0. \tag{4}$$

By construction the map $\mathbb{Z}G^m \to \mathbb{Z}\overline{E}$ factors through a surjection $\mathbb{Z}G^m \to H_1(N;\mathbb{Z})$. Therefore, the exactness of (3) ensures the exactness of (4). It follows that $Y$ is 1-acyclic.

Let $V_0$ and $E_0$ be sets of orbit representatives of vertices and edges in $Y$. Now, applying Proposition 3.1 then (2), we have that

$$\begin{aligned}\chi^{(2)}(F) + m &= \sum_{v\in V_0} \frac{1}{|G_v|} - \sum_{e\in E_0} \frac{1}{|G_e|} + m \\ &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y).\end{aligned}$$

Lemma 2.1 with $n = 2$, yields

$$\begin{aligned}\chi^{(2)}(F) + m &= b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y) + b_2^{(2)}(EG \times Y) \\ &\geq b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y).\end{aligned}$$

Applying [12, Theorem 6.54(1a)] to the projection $f : EG \times Y \to EG$ with $n = 2$ (note that we are using the fact $Y$ is 1-acyclic), we obtain $b_i^{(2)}(EG \times Y) = b_i^{(2)}(EG)$ for $i = 0$, 1. Recalling (1), we therefore have

$$\chi^{(2)}(F) + m \geq b_0^{(2)}(G) - b_1^{(2)}(G).$$

Let $k = \chi^{(2)}(F) + m$. If $k \leq 0$, then $b_0^{(2)}(G) - b_1^{(2)}(G) \leq 0$ and so $G$ is infinite, this proves (i). Now, assume $k < 0$. In this case, $G$ is infinite and therefore $b_0^{(2)}(G) = 0$. It follows that $b_1^{(2)}(G) \geq -k > 0$, this proves (ii).

If $G$ is finite, then $b_0^{(2)}(G) = \frac{1}{|G|}$, $b_1^{(2)}(G) = 0$, and $k > 0$. In particular, $k \geq \frac{1}{|G|} > 0$ and (iii) follows. $\square$

## 4. On the $\ell^2$-invariants for certain groups acting on trees

**Proof of Theorem 1.5.** Let $V$ and $E$ denote sets of representatives of $F$-orbits of vertices and edges for the action of $F$ on the tree. We consider the relevant part of the $E^1$-page for the $F$-equivariant spectral sequence (see Chapter VII.9 of [6]) applied to the tree:

$$\begin{array}{c|cc} 1 & \bigoplus_{v\in V} H_1^F(F \times_{F_v} EF_v; \mathscr{U}F) & 0 \\ 0 & \bigoplus_{v\in V} H_0^F(F \times_{F_v} EF_v; \mathscr{U}F)) \xleftarrow{\;d^1\;} & \bigoplus_{e\in E} H_0^F(F \times_{F_e} EF_e; \mathscr{U}F)) \\ \hline & 0 & 1 \end{array}$$

If $F$ is finite then $b_1^{(2)}(F) = 0$, so $d^1$ is injective and $E^2_{1,0} = 0$. The result follows from the fact $E^1_{0,1} = 0$.

Now, assume $F$ is infinite, then $d^1$ is surjective since $b_0^{(2)}(F) = 0$. Thus,

$$\dim_{\mathcal{U}F}(\mathrm{Ker}(d^1)) = \sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v).$$

Now, the spectral sequence obviously collapses on the $E^2$-page and $E^1_{0,1} = E^2_{0,1}$. Since von Neumann dimension is additive over short exact sequences, we have

$$\begin{aligned} b_1^{(2)}(F) &= \dim_{\mathcal{U}F}(\mathrm{Ker}(d^1)) + \dim_{\mathcal{U}F}(E^2_{0,1}) \\ &= \left(\sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v)\right) + \sum_{v\in V} b_1^{(2)}(F_v), \end{aligned}$$

and the result follows. □

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