Geometriae Dedicata **96:** 161–177, 2003.

Property (RD) for Cocompact Lattices in a Finite Product of Rank One Lie Groups with Some Rank Two Lie Groups

INDIRA CHATTERJI
Department of Mathematics, Cornell University, Ithaca, NY 14863, U.S.A.
e-mail: indira@math.cornell.edu

(Received: 9 April 2001; in final form: 21 January 2002)

Abstract. We apply V. Lafforgue's techniques to establish property (RD) for cocompact lattices in a finite product of rank one Lie groups with Lie groups whose restricted root system is of type A_2.

Mathematics Subject Classification (2000). 22E40.

Key words. rapid decay, Baum–Connes conjecture, Lie groups.

We recall that a *length function* on a discrete group Γ is a function $\ell\colon \Gamma \to \mathbf{R}_+$ such that the neutral element is mapped to zero, and such that $\ell(\gamma) = \ell(\gamma^{-1})$, $\ell(\gamma\mu) \leqslant \ell(\gamma) + \ell(\mu)$ for any $\gamma, \mu \in \Gamma$. A discrete group Γ is said to have *property* (*RD*) *with respect to* ℓ if there exists a polynomial P such that for any $r \in \mathbf{R}_+$ and $f \in \mathbf{C}\Gamma$ supported on elements of length shorter than r the inequality $\|f\|_* \leqslant P(r)\|f\|_2$ holds, where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(\Gamma)$, and $\|f\|_2$ the usual ℓ^2 norm. Property (RD) was first established for free groups by Haagerup [5], but introduced and studied by Jolissaint [8], who established it for classical hyperbolic groups. The extension to Gromov hyperbolic groups is due to P. de la Harpe [6]. Providing the first examples of higher rank groups, J. Ramagge, *et al.* [12] proved that property (RD) holds for discrete groups acting freely on the vertices of an $\tilde{A}_1 \times \tilde{A}_1$ or $\tilde{A}_2$ building and, recently, V. Lafforgue did so for cocompact lattices in $SL_3(\mathbf{R})$ and $SL_3(\mathbf{C})$ in [9]. Cocompactness is crucial since the only (up to now) known obstruction to property (RD) has been given by Jolissaint [8] and in the presence of an amenable subgroup with exponential growth, which according to Lubotzky *et al.* [11] occurs in any noncocompact lattice in higher rank. A conjecture, due to Valette, (see [2]), claims that the property (RD) holds for any discrete group acting isometrically, properly and cocompactly either on a Riemannian symmetric space or on an affine building. More discussion concerning the property (RD) can be found in [1]. Property (RD) is important in the context of the Baum–Connes conjecture, precisely, Lafforgue [10] proved that for 'good' groups having the property (RD), the Baum–Connes conjecture without coefficients holds.

In this article we establish property (RD) for discrete cocompact subgroups Γ of a finite product of type $\mathrm{Iso}(\mathcal{X}_1) \times \cdots \times \mathrm{Iso}(\mathcal{X}_n)$, where the $\mathcal{X}_i$'s are either proper Gromov hyperbolic spaces, buildings associated to $SL_3(F)$ (for F a non-Archimedean locally compact field), or symmetric spaces associated to Lie groups whose restricted root system is of type A_2 (those are known to be locally isomorphic to $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$, see [7]). In particular, this immediately implies the following theorem:

THEOREM 0.1. *Any cocompact lattice in* $G = G_1 \times \cdots \times G_n$ *has property* (*RD*), *where the* G_i*'s are either rank one Lie groups,* $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ *or* $E_{6(-26)}$.

To establish this result, we will first answer (positively) a question posed by V. Lafforgue in [9], which was to know whether his lemmas 3.5 and 3.7 are still true for the groups $SL_3(\mathbf{H})$ and $E_{6(-26)}$, whose associated symmetric spaces also have flats of type A_2. Observing that these lemmas are in fact 'three points conditions' it will be enough to prove that if X denotes $SL_3(\mathbf{H})/SU_3(\mathbf{H})$ or $E_{6(-26)}/F_{4(-52)}$, then for any three points in X, there exists a totally geodesic embedding of $SL_3(\mathbf{C})/SU_3(\mathbf{C})$ containing those three points. Secondly, we will explain how to use the techniques used in [12] and [9] for the above described products. This provides many new examples of groups with property (RD), such as irreducible cocompact lattices in $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$, and irreducible cocompact lattices in semisimple Lie groups of arbitrary rank (obtained, for instance, with products of rank one Lie groups).

Finally, we remark that combining our result with Lafforgue's crucial theorem in [10] yields the following

COROLLARY 0.2. *The Baum–Connes conjecture without coefficients* (*see* [16]) *holds for any cocompact lattice in* $G = G_1 \times \cdots \times G_n$, *where the* G_i*'s are either rank one Lie groups,* $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ *or* $E_{6(-26)}$.

1. The Case of $SL_3(H)$

We will write $\mathbf{H}$ for *Hamilton's Quaternion algebra*, which is a four dimensional vector space over $\mathbf{R}$, whose basis is given by the elements $1, i, j$ and k, satisfying

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ki = j, \quad jk = i.$$

This is an associative division algebra endowed with an involution $h \mapsto \bar{h}$ which is the identity over 1, and minus the identity over i, j and k. A norm on $\mathbf{H}$ can be given by $|h| = \sqrt{h\bar{h}} \in \mathbf{R}_+$. A quaternion will be called a *unit* if of norm one, *real* if lying in $\mathrm{span}\{1\}$ and *imaginary* if lying in $\mathrm{span}\{1\}^{\perp}$ (for the scalar product of $\mathbf{R}^4$ which turns the above described basis in an orthonormal basis). Note that an imaginary unit has square -1. The following lemma is obvious:

LEMMA 1.1. *Any element $h \in \mathbf{H}$ is contained in a commutative subfield of* $\mathbf{H}$.

DEFINITION 1.2. Denote by I the identity matrix in $M_3(\mathbf{C})$, and

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_6(\mathbf{C}),$$

we write

$$\begin{aligned} M_3(\mathbf{H}) &= \{M \in M_6(\mathbf{C}) \text{ such that } JMJ^{-1} = \bar{M}\}, \\ SL_3(\mathbf{H}) &= \{M \in M_3(\mathbf{H}) \text{ such that } \det(M) = 1\}, \\ SU_3(\mathbf{H}) &= \{M \in SL_3(\mathbf{H}) \text{ such that } MM^* = M^*M = I\}. \end{aligned}$$

Remark 1.3. (1) Given any imaginary unit $\mu \in \mathbf{H}$, we can write $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}\mu$, and thus decompose any $h \in \mathbf{H}$ as $h = h_1 + h_2\mu$, with $h_1, h_2 \in \mathbf{C}$. Similarly, any 3×3 matrix M with coefficients in $\mathbf{H}$ can be written $M = M_1 + M_2\mu$, with $M_1, M_2 \in M_3(\mathbf{C})$. The $*$-homomorphism

$$M \mapsto \begin{pmatrix} M_1 & -\bar{M}_2 \\ M_2 & \bar{M}_1 \end{pmatrix}$$

gives then an isomorphism between the algebra of 3×3 matrices with coefficients in $\mathbf{H}$ (and usual multiplication) and $M_3(\mathbf{H})$ as just defined. It is a straightforward computation to see that elements of type

$$\mathcal{T}_\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with λ a unit in $\mathbf{H}$ actually belong to $SU_3(\mathbf{H})$.

DEFINITION 1.4. Let $\mathbf{K}$ denote $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. We set

$$X_{\mathbf{K}} = SL_3(\mathbf{K})/SU_3(\mathbf{K}).$$

It will be convenient for us to choose

$$\{M \in SL_3(\mathbf{K}) \text{ such that } M^* = M, M \text{ positive}\}$$

as a model for $X_{\mathbf{K}}$. On $X_{\mathbf{K}}$ we consider the action of $SL_3(\mathbf{K})$ given by

$$\begin{aligned} SL_3(\mathbf{K}) \times X_{\mathbf{K}} &\to X_{\mathbf{K}} \\ (g, z) &\mapsto g(z) = (gz^2g^*)^{1/2}. \end{aligned}$$

This action is transitive since for $M \in X_{\mathbf{K}}$, setting $g = M$ we get that $g(I) = M$. Moreover, the stabilizer of I is clearly $SU_3(\mathbf{K})$. We equip $X_{\mathbf{K}}$ with the distance

$$d_{\mathbf{K}}(x, y) = \log \|x^{-1}y\| + \log \|y^{-1}x\|,$$

where $\| \ \|$ denotes the operator norm on $SL_3(\mathbf{K})$ acting on $\mathbf{K}^3$. Notice that for $a \in X_{\mathbf{K}}$ a diagonal matrix (thus real), if we assume that

$$a = \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} \quad \text{with } \alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 0,$$

we get that $d_{\mathbf{K}}(a, I) = \alpha_1 - \alpha_3$. Denote by A the set of all diagonal matrices in $X_{\mathbf{K}}$. To stick with the classical terminology, we will call *flats* the sets of type $g(A)$ for a $g \in SL_3(\mathbf{K})$.

Remark 1.5. The action of $SL_3(\mathbf{K})$ on $X_{\mathbf{K}}$ is by isometries with respect to the above given distance (in other words, the distance is $SL_3(\mathbf{K})$-invariant): For $x, y \in X_{\mathbf{K}}$, we have to show that the operator norm of $x^{-1}y$ is equal to that of

$$z = g(y)^{-1}g(x) = (gx^2g^*)^{-1/2}(gy^2g^*)^{1/2}.$$

But for any $k_1, k_2 \in SU_3(\mathbf{K})$, $\|z\| = \|k_1zk_2\|$, so setting

$$k_1 = (gx)^{-1}(gx^2g^*)^{1/2} \quad \text{and} \quad k_2 = (gy^2g^*)^{-1/2}gy,$$

we get that $k_1zk_2 = x^{-1}y$. Notice that for $g \in SL_3(\mathbf{K})$, the standard action $z \mapsto gzg^*$ is not isometric. Indeed, the operator norm is in general not invariant by conjugation by g unless $g = k \in SU_3(\mathbf{K})$, and in that case

$$k(z) = (kz^2k^*)^{1/2} = (kzk^*kzk^*)^{1/2} = kzk^*.$$

LEMMA 1.6. *Let* φ: $\mathbf{C} \to \mathbf{H}$ *be an isometric injective ring homomorphism. It induces a group homomorphism* φ^*: $SL_3(\mathbf{C}) \to SL_3(\mathbf{H})$ *which induces a totally geodesic isometric embedding* $\bar{\varphi}$: $X_{\mathbf{C}} \to X_{\mathbf{H}}$.

Proof. That φ^* is a homomorphism is clear in view of Remark 1.3, so let us now show that $\bar{\varphi}$ is an isometry. Since $SL_3(\mathbf{C})$ acts transitively and by isometries on $X_{\mathbf{C}}$, it will be enough to show that $d_{\mathbf{H}}(\bar{\varphi}(x), I) = d_{\mathbf{C}}(x, I)$ for any $x \in X_{\mathbf{C}}$. Noticing that $\varphi^*(SU_3(\mathbf{C})) \subset SU_3(\mathbf{H})$ we deduce that for any $g \in SL_3(\mathbf{C})$, $\bar{\varphi}, (g(x)) = \varphi^*(g)(\bar{\varphi}(x))$. Now, for $x \in X_{\mathbf{C}}$, there exists $k \in SU_3(\mathbf{C})$ such that $k(x) = a$ is diagonal with positive coefficients. Since φ is a ring homomorphism, it is the identity over the real numbers and so:

$$\begin{aligned} d_{\mathbf{C}}(x, I) &= d_{\mathbf{C}}(k(x), k(I)) = d_{\mathbf{C}}(a, I) = d_{\mathbf{H}}(a, I) = d_{\mathbf{H}}(\varphi^*(k^*)(a), \varphi^*(k^*)(I)) \\ &= d_{\mathbf{H}}(\bar{\varphi}(k^*(a)), I) = d_{\mathbf{H}}(\bar{\varphi}(x), I). \end{aligned}$$

Let us now prove that the embedding is totally geodesic. Take x, y in the image of $\bar{\varphi}$, so that $x = \bar{\varphi}(x')$ and $y = \bar{\varphi}(y')$ for $x', y' \in X_{\mathbf{C}}$. There is a $g \in SL_3(\mathbf{C})$ so that

$x' = g(I)$ and $y' = g(a)$ for $a \in A$. Then $x = \bar{\varphi}(g(I)) = \varphi^*(g)(I)$ and $y = \bar{\varphi}(g(a)) = \varphi^*(g)(a)$. Since A is obviously in the image of $\bar{\varphi}$, so will $\varphi^*(g)(A)$ be (that is, the whole flat containing x and y is in the image of the embedding). Since a geodesic γ between x and y lies in any flat containing them, we conclude that $\gamma \subset \varphi^*(g)(A) \subset \bar{\varphi}(X_{\mathbf{C}})$. □

Remark 1.7. Any element of $X_{\mathbf{H}}$ can be diagonalized using elements of type $\mathcal{T}_h$ (described in Remark 1.3) and of $SO_3(\mathbf{R})$. Indeed, take $z = (z_{ij}) \in X_{\mathbf{H}}$, then $z' = \mathcal{T}_h(z)$ with $h = \overline{z_{12}}/|z_{12}|$ verifies that $z'_{ij} \in \mathbf{R}$ for $i, j = 1, 2$ (in case $z_{12} = 0$, or if it lies in $\mathbf{R}$ we can just skip this step). Take $k \in SO_2(\mathbf{R})$ diagonalizing the 2×2 matrix (z'_{ij}) $i, j = 1, 2$. Then

$$z'' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} z' \begin{pmatrix} k^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} z''_{11} & z''_{12} & z''_{13} \\ \overline{z''_{12}} & z''_{22} & 0 \\ \overline{z''_{13}} & 0 & z''_{33} \end{pmatrix}$$

with $z''_{11}, z''_{22}, z''_{33} \in \mathbf{R}$. Repeating this argument twice we get the diagonalization.

The above given argument shows that the subgroup of $SL_3(\mathbf{H})$ generated by the $\mathcal{T}_h$'s and $SL_3(\mathbf{H})$ acts transitively on $X_{\mathbf{H}}$. Since the $\mathcal{T}_h$'s and $SO_3(\mathbf{R})$ do stabilize I, we can conclude that $SL_3(\mathbf{H})$ is generated by elements of type $\mathcal{T}_h$ and $SL_3(\mathbf{H})$.

PROPOSITION 1.8. *For any three points in $X_{\mathbf{H}}$ there is a totally geodesic embedding of $X_{\mathbf{C}}$ containing those three points.*

Proof. Let $x, y, z \in X_{\mathbf{H}}$. Up to multiplication by an element of $SL_3(\mathbf{H})$ (which is an isometry), we can assume that $x = I$ and y is a diagonal matrix (see the previous Remark). We write

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ \overline{z_{12}} & z_{22} & z_{23} \\ \overline{z_{13}} & \overline{z_{23}} & z_{33} \end{pmatrix}$$

with $z_{11}, z_{22}, z_{33} \in \mathbf{R}$ and z_{12}, z_{13} and z_{23} in $\mathbf{H}$. Consider the element

$$k = \begin{pmatrix} \overline{z_{12}}/|z_{12}| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{23}/|z_{23}| \end{pmatrix} \in SU_3(\mathbf{H}),$$

then $k(z)$ only has real elements except for $k(z)_{13} = \overline{k(z)_{31}}$, and $k(a) = a$ for any diagonal element a. Applying Lemma 1.1 to $k(z)_{13}$, we get an embedding of $\mathbf{C}$ in $\mathbf{H}$ which contains $k(z)_{13}$, and thus a totally geodesical embedding $\bar{\varphi}$: $X_{\mathbf{C}} \to X_{\mathbf{H}}$ whose image contains x, y and z. □

2. The Case of $E_{6(-26)}$

We will write **O** for the eight-dimensional algebra over **R**, whose basis is given by the elements $e_0, e_1, \ldots, e_7$, satisfying

$$e_ie_0 = e_0e_i = e_i, e_i^2 = -e_0, \quad \text{for all } i = 1, \ldots, 7,$$
$$e_ie_j = -e_je_i, \quad \text{if } i \neq j \text{ and } i, j \neq 0,$$
$$e_2e_6 = e_3e_4 = e_5e_7 = e_1$$

and all those one can deduce by cyclically permuting the indices from 1 to 7. This is a nonassociative division algebra, endowed with an involution $x \mapsto \bar{x}$ which is the identity over e_0, and minus the identity over e_i, for all $i = 1, \ldots, 7$. A norm on **O** can be given by $|x| = \sqrt{x\bar{x}} \in \mathbf{R}_+$. The division algebra **O** is called the *Cayley Octonions*. An octonion will be called a *unit* if of norm one, *real* if lying in span$\{e_0\}$ and *imaginary* if lying in span$\{e_0\}^\perp$ (for the scalar product of $\mathbf{R}^8$ which turns the above described basis in an orthonormal basis). The following theorem:

THEOREM 2.1 (Artin, see [13] p. 29). *Any two elements of* **O** *are contained in an associative sub-division algebra of* **O***,*

will allow us to apply the arguments used in the previous section to $E_{6(-26)}$.

We will now give a definition of $E_{6(-26)}$, of a maximal compact subgroup $F_{4(-52)}$, and of a model for $X_\mathbf{O} = E_{6(-26)}/F_{4(-52)}$. This part is based on the work of H. Freudenthal, see [3] and [4].

DEFINITION 2.2. Denote by $M_3(\mathbf{O})$ the set of 3×3 matrices with coefficients on **O**. For $M = (m_{ij}) \in M_3(\mathbf{O})$ write $M^* = \bar{M}^t = (\bar{m}_{ji})$. The exceptional Jordan algebra is given by

$$\mathcal{J} = \{M \in M_3(\mathbf{O}) \text{ such that } M = M^*\}$$

which is stable under the *Jordan multiplication* given by $M \star N = \frac{1}{2}(MN + NM)$. H. Freudenthal defined an application

$$\det\colon \mathcal{J} \to \mathbf{R}$$
$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \xi_1\xi_2\xi_3 - \left(\sum_{i=1}^{3} \xi_i |x_i|^2\right) + 2\Re(x_1x_2x_3)$$

(where $\Re$ denotes the real part, and is well defined even without parenthesis), showed that

$$E_{6(-26)} = \{g \in GL(\mathcal{J}) \text{ such that } \det \circ g = \det\},$$

and that

$$F_{4(-52)} = \{g \in E_{6(-26)} \text{ such that } g(I) = I\}$$

is the automorphism group of $\mathcal{J}$ and is a maximal compact subgroup in $E_{6(-26)}$.

Remark 2.3. We will now explicitly show some elements of $E_{6(-26)}$ (see [4] for the proofs of their belonging to $E_{6(-26)}$):

(1) Any element x of $SL_3(\mathbf{R})$ gives a map $x\colon \mathcal{J} \to \mathcal{J}$ by $M \mapsto x(M) = xMx^t$ which preserves the determinant.
(2) Let a be a unit in $\mathbf{O}$, define

$$\varphi_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \bar{a} \end{pmatrix}.$$

The map $\psi_a\colon \mathcal{J} \to \mathcal{J}$, defined as $\psi_a(M) = \varphi_{\bar{a}} M \varphi_a$ (and no parenthesis are needed) is in $E_{6(-26)}$, and even in $F_{4(-52)}$.
(3) Freudenthal proves (see [3], p. 40) that elements in $\mathcal{J}$ are diagonalizable by elements in $F_{4(-52)}$, that the elements on the diagonal are uniquely determined up to permutation.

DEFINITION 2.4. We say that an element $M \in \mathcal{J}$ is *positive* if after diagonalization it only has positive elements. We define

$$X_{\mathbf{O}} = \{M \in \mathcal{J} \text{ such that } \det(M) = 1 \text{ and positive}\}.$$

We let $E_{6(-26)}$ act on $X_{\mathbf{O}}$ as follows:

$$\begin{aligned} E_{6(-26)} \times X_{\mathbf{O}} &\to X_{\mathbf{O}} \\ (g, M) &\mapsto g \circ M = \sqrt{g(M^2)} \end{aligned}$$

where (for M positive) $\sqrt{M}$ is an element Y in $X_{\mathbf{O}}$ satisfying $Y \star Y = M$, and well defined by requiring positivity. Notice that for $k \in F_{4(-52)}$, we have that $k \circ M = k(M)$.

PROPOSITION 2.5. *The group* $E_{6(-26)}$ *acts transitively on* $X_{\mathbf{O}}$, *and the stabilizer of* I *is* $F_{4(-52)}$.

Proof. Since $k \circ I = k(I) = I$, the stabilizer of I is $F_{4(-52)}$, and the action is transitive because given any $M \in X_{\mathbf{O}}$, there exists a $k \in F_{4(-52)}$ such that $k \circ M = D$ is diagonal. Since $\det(M) = \det(k(M)) = 1$, the element D^{-1} defines an element in $E_{6(-26)}$ (coming from $SL_3(\mathbf{R})$), and $D^{-1} \circ D = I$. □

Remark 2.6. For any two elements M, N of $X_{\mathbf{O}}$, there exists a $g \in E_{6(-26)}$ such that $M = g \circ I$ and $N = g \circ D$ where D is a diagonal matrix in $X_{\mathbf{O}}$. Indeed, in the proof of the preceding proposition we saw that there exists an $h \in E_{6(-26)}$ with $I = h \circ M$, and that any matrix (and thus in particular $h \circ N$) is diagonalizable with elements of $F_{4(-52)}$. We then choose $k \in F_{4(-52)}$ such that $k(h \circ N) = D$ is diagonal, and set $g = (kh)^{-1}$.

DEFINITION 2.7. For a diagonal element D in $X_{\mathbf{O}}$, we set

$$\|D\| = \max\{d_i | i = 1, 2, 3\}$$

where the d_i's denote the elements in the diagonal of D. We equip $X_{\mathbf{O}}$ with the distance

$$d(M, N) = \log \|D\| + \log \|D^{-1}\|.$$

for D as in the previous remark. It is well defined because of point 3 of Remark 2.3.

Any embedding of $\mathbf{H}$ in $\mathbf{O}$ gives a decomposition $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}\ell$ (where ℓ is any unit in the orthogonal complement of the embedded copy of $\mathbf{H}$ in $\mathbf{O}$), and we will now describe some other elements in $F_{4(-52)}$ that will later be useful to embed $SL_3(\mathbf{H})$ in $E_{6(-26)}$ starting from an embedding of $\mathbf{H}$ in $\mathbf{O}$. Given a unit $h \in \mathbf{H}$ we define two maps

$$\begin{aligned} &t_h, u_h : \mathbf{O} \to \mathbf{O} \\ &x = a + b\ell \mapsto t_h(x) = ha + b\ell \\ &\qquad\qquad\quad u_h(x) = a + (b\bar{h})\ell \end{aligned}$$

and an element (that we will later prove to belong to $F_{4(-52)}$)

$$T_h \colon \mathcal{J} \to \mathcal{J}$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & t_h(x_3) & t_h(\bar{x}_2) \\ \overline{t_h(x_3)} & \xi_2 & u_h(\bar{x}_1) \\ \overline{t_h(\bar{x}_2)} & u_h(\bar{x}_1) & \xi_3 \end{pmatrix}$$

Using that for $x = a + b\ell$ and $y = c + d\ell$, the product xy reads $(ac - \bar{d}b) + (b\bar{c} + da)\ell$ (see [14]), it is a direct computation to check that $\det \circ T_h = \det$.

Knowing that $T_h \in E_{6(-26)}$, it is pretty clear that $T_h(I) = I$, and thus $T_h \in F_{4(-52)}$. We will now, for any given embedding of $\mathbf{H}$ in $\mathbf{O}$, define an explicit embedding of $SL_3(\mathbf{H})$ in $E_{6(-26)}$.

LEMMA 2.8. *Let S denote the subgroup of $E_{6(-26)}$ generated by elements of the type $x \in SL_3(\mathbf{H})$ and T_h for h a unit in $\mathbf{H}$ as just described. Then S is isomorphic to $SL_3(\mathbf{H})$.*

Proof. An embedding of $\mathbf{H}$ in $\mathbf{O}$ defines an embedding of $X_{\mathbf{H}}$ in $\mathcal{J}$ by taking the images in $\mathbf{O}$ of the coefficients of an element in $X_{\mathbf{H}}$. It will now be enough to see that the map

$$\begin{aligned} &\rho \colon S \to SL_3(\mathbf{H}) \\ &\quad z \mapsto z|_{X_{\mathbf{H}}} \end{aligned}$$

is well defined and a group isomorphism.

Remember that for h a unit in $\mathbf{H}$ we denoted by $\mathcal{T}_h \in SL_3(\mathbf{H})$ a diagonal matrix with h in the first place and 1's in the second and third place. Now, notice (from the definition of $\mathcal{T}_h$) that $\rho(T_h) = \mathcal{T}_h$ for any h a unit in $\mathbf{H}$ and that trivially, for

$x \in SL_3(\mathbf{R})$, $\rho(x) \in SL_3(\mathbf{H})$. This shows that the image of ρ is contained in $SL_3(\mathbf{H})$. The map ρ being a restriction, it is a group homomorphism.

Our map ρ is clearly injective, and because of Remark 1.5 we know that ρ maps the generators of S onto those of $SL_3(\mathbf{H})$. □

PROPOSITION 2.9. *For any three points in $X_{\mathbf{O}}$ there is a totally geodesic embedding of $X_{\mathbf{C}}$ containing those three points.*

Proof. Let $x, y, z \in X_{\mathbf{O}}$. Up to multiplication by an element of $E_{6(-26)}$ (which is an isometry), we can assume that $x = I$ and y is a diagonal matrix (use Remark 2.6). We write

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ \overline{z_{12}} & z_{22} & z_{23} \\ \overline{z_{13}} & \overline{z_{23}} & z_{33} \end{pmatrix}$$

with $z_{11}, z_{22}, z_{33} \in \mathbf{R}$ and z_{12}, z_{13} and z_{23} in $\mathbf{O}$. Consider the element $k = \phi_a$ where $a = \overline{z_{12}}/|z_{12}|$ is a unit in $\mathbf{O}$ then $k(z)$ only has real elements except for $k(z)_{13} = \overline{k(z)_{31}}$, and $k(d) = d$ for any diagonal element in $X_{\mathbf{O}}$. Applying Theorem 2.1 to $k(z)_{13}$ and $k(z)_{23}$, we get an embedding of $\mathbf{H}$ in $\mathbf{O}$ which contains $k(z)_{13}$ and $k(z)_{23}$, and thus an embedding $\bar{\varphi}$: $X_{\mathbf{H}} \to X_{\mathbf{O}}$ whose image contains x, y and z, and which is isometric by definition of the distance on $X_{\mathbf{O}}$.

To see that the embedding is totally geodesic is basically the same argument as in the proof of Proposition 1.6: Take x, y in the image of $\bar{\varphi}$, so that $x = \bar{\varphi}(x')$ and $y = \bar{\varphi}(y')$ for $x', y' \in X_{\mathbf{C}}$. There is a $g \in SL_3(\mathbf{C})$ so that $x' = g(I)$ and $y' = g(a)$ for $a \in A$. Now, the embedding $\bar{\varphi}$ comes from an embedding $\varphi*$: $SL_3(\mathbf{H}) \to E_{6(-26)}$ described in Lemma 2.8, so that $x = \bar{\varphi}(g(I)) = \varphi^*(g)(I)$ and $y = \bar{\varphi}(g(a)) = \varphi^*(g)(a)$. Since A is obviously in the image of $\bar{\varphi}$, so will $\varphi^*(g)(A)$ be (that is, the whole flat containing x and y is in the image of the embedding). Since a geodesic γ between x and y lies in any flat containing them, we conclude that $\gamma \subset \varphi^*(g)(A) \subset \bar{\varphi}(X_{\mathbf{C}})$.

We are now reduced to find an embedding of $X_{\mathbf{C}}$ in $X_{\mathbf{H}}$ containing x, y and z, but we get this one because of Proposition 1.8, and combining it with $\bar{\varphi}$ we get the sought embedding. □

3. How We Answer V. Lafforgue's Question

We will now see how Proposition 1.8 and 2.9 imply that Lemmas 3.5 and 3.7 of Lafforgue's paper [9] hold. To see how property (RD) can be deduced we refer to the next section. But before proceeding, here are some definitions and notations.

DEFINITION 3.1. Let G denote $SL_3(\mathbf{K})$ (for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) or $E_{6(-26)}$ and K denote $SU_3(\mathbf{K})$ or $F_{4(-52)}$, and $X_{\mathbf{K}} = G/K$. Denote by A the diagonal matrices in $X_{\mathbf{K}}$. The Cartan decomposition states then that $G = K(A)$. Moreover, for any $x, y \in X_{\mathbf{K}}$, there exists $g \in G$ so that $x, y \in g(A)$. For any $t \in \mathbf{R}$, $x, y, z \in X_{\mathbf{K}}$, we say that (x, y) is *of shape* $(t, 0)$ if there exists $g \in G$ such that

$$g(x) = I, \qquad g(y) = e^{-t/3}\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that (x, y, z) is an *equilateral triangle* of oriented size t (where $t \in \mathbf{R}$ can be positive or negative) if there exists $g \in G$ such that

$$g(x) = I, \qquad g(y) = e^{-t/3}\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g(z) = e^{-2t/3}\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let (X, d) be a metric space and $\delta \geqslant 0$. For any finite sequence of points $x_1 \ldots, x_n \in X$ we say that $x_1 \ldots x_n$ is a *δ-path* if

$$d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \leqslant d(x_1, x_n) + \delta$$

and that three points $x, y, z \in X$ form *δ-retractable triple* if there exists $t \in X$ such that the paths xty, ytz and ztx are δ-paths.

LEMMA 3.2. *Let* $\mathbf{K} = \mathbf{H}$ *or* $\mathbf{O}$. *For any* $\delta_0 > 0$ *there exists a* $\delta > 0$ *such that the following is true:*

(1) *For any* $x, y, z \in X_{\mathbf{K}}$ *there exists* x', y', z' *an equilateral triangle in* $X_{\mathbf{K}}$ *such that the paths* $xx'y'y$, $yy'z'z$, $zz'x'x$ *are δ-paths.*
(2) *For any* $s, t \in \mathbf{R}$ *of same sign and* $x, v, w, y \in X_{\mathbf{K}}$ *such that* $d(v, w) < \delta_0$, (v, z) *is of shape* $(t, 0)$ *and* (w, y) *is of shape* $(s, 0)$, *the triangle* z, v, y *is δ-retractable.*

Proof. (1) Because of Propositions 1.8 and 2.9, there is an isometric copy of $X_{\mathbf{C}}$ in $X_{\mathbf{K}}$ containing x, y, z. Thus, because of Lemma 3.6 in [9] there exists $\delta \geqslant 0$ and x', y', z' an equilateral triangle in $X_{\mathbf{C}}$ such that the paths $xx'y'y$, $yy'z'z$ and $zz'x'x$ are δ-paths. Since the embedding is totally geodesic, the triangle x', y', z' will be equilateral in $X_{\mathbf{K}}$ as well.

(2) Without loss of generality, we can assume that $w = I$ and that y is diagonal, so that $z, v \in h(A)$ for an $h \in G$ such that $d(h(I), I) \leqslant \delta_0$. Because of Proposition 1.8 and 2.9 there is a totally geodesic embedding $\bar{\varphi}\colon X_{\mathbf{C}} \to X_{\mathbf{K}}$ containing $h(I)$, w and y. But since $\bar{\varphi}$ comes from an embedding $\varphi\colon SL_3(\mathbf{C}) \to G$ we have that:

$$h(I) = \bar{\varphi}(\bar{h}) = \bar{\varphi}(h'(I)) = \varphi(h')(I)$$

(for an $\bar{h} \in X_{\mathbf{C}}$ and $h' \in SL_3(\mathbf{C})$) so that the whole $h(A)$ is contained in the image of $\bar{\varphi}$ in $X_{\mathbf{K}}$ and we thus can see x, v, w and y in $X_{\mathbf{C}}$. Using Lemma 3.7 in [9], the triangle z, v, y is δ-retractable in $X_{\mathbf{C}}$, and since the embedding is totally geodesic, it will be δ-retractable in $X_{\mathbf{K}}$ as well. □

4. How to Establish Property (RD)

We start by recalling some definitions that can be found in [9].

DEFINITION 4.1. Let $\delta \geqslant 0$. A discrete metric space (X, d) satisfies *property* (H_δ) if for there exists a polynomial P_δ such that for any $r \in \mathbf{R}_+$, $x, y \in X$ one has

$$\sharp\{t \in X \text{ such that } d(x, t) \leqslant r, \text{ and } xty \text{ is a } \delta - \text{path}\} \leqslant P_\delta(r).$$

We say that (X, d) satisfies *property* (H) if it satisfies (H_δ) for any $\delta \geqslant 0$. For any $r \in \mathbf{R}_+$, and $x \in X$ let $B(x, r) \subset X$ denote the open ball of radius r centered in x. A subset $Y \subset X$ is a *uniform net* in X if there exists two constants r_Y and R_Y in $\mathbf{R}_+$ such that

$$B(y, r_Y) \cap Y = \{y\}, \quad \text{for any } y \in Y,$$
$$B(x, R_Y) \cap Y \neq \emptyset, \quad \text{for any } x \in X.$$

Remark 4.2. In other words, property (H) gives a polynomial bound for geodesics between two points. V. Lafforgue, in [9] proved that any uniform net in a semi-simple Lie group has property (H), provided that the distance takes into account every root.

DEFINITION 4.3. Let (X, d) be a metric space and Γ be a discrete group acting by isometries on X. The pair (X, Γ) satisfy *property* (K) if there exists $\delta \geqslant 0$, $k \in \mathbf{N}$ and Γ-invariant subsets $\mathcal{T}_1, \ldots, \mathcal{T}_k$ of X^3 such that

(K_a) There exists $C \in \mathbf{R}_+$ such that for any $(x, y, z) \in X^3$, there exists $i \in \{1, \ldots, k\}$ and $(\alpha, \beta, \gamma) \in \mathcal{T}_i$ such that $\max\{d(\alpha, \beta), d(\beta, \gamma), d(\gamma, \alpha)\} \leqslant C \min\{d(x, y), d(y, z), d(z, x)\} + \delta$ and $x\alpha\beta y$, $y\beta\gamma z$, $z\gamma\alpha x$ are δ-paths.

(K_b) For any $i \in \{1, \ldots, k\}$ and $\alpha, \beta, \gamma, \gamma' \in X$, if $(\alpha, \beta, \gamma) \in \mathcal{T}_i$ and $(\alpha, \beta, \gamma') \in \mathcal{T}_i$ then the triangles $\alpha\gamma\gamma'$ and $\beta\gamma\gamma'$ are δ-retractable.

THEOREM 4.4 (V. Lafforgue [9]). *Let X be a discrete metric space, and let Γ be a group acting freely and isometrically on X. If the pair (X, Γ) satisfies (H) and (K), then Γ satisfies property (RD).*

From now on, $\mathcal{K}$ will denote a finite product of metric spaces as described in the introduction, endowed with the ℓ^1 combination of the distances. Namely

$$(\mathcal{K}, d) = (\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, d_1 + \cdots + d_n),$$

where for $i = 1, \ldots, n$ the $(\mathcal{X}_i, d_i)$'s are either Gromov hyperbolic spaces, buildings associated to $SL_3(F)$ (for F a non-archimedean locally compact field) endowed with the graph theoretical distance associated to the one skeleton (as in [12]), or symmetric spaces associated to $SL_3(\mathbf{R})$, $SL_3(\mathbf{C})$, $SL_3(\mathbf{H})$ and $E_{6(-26)}$ endowed with the Finsler norm described previously. The two following lemmas will show why any uniform net in $\mathcal{K}$ has property (H).

LEMMA 4.5. *Let X be a uniform net in a geodesic metric space $(\mathcal{H}, d)$. If X satisfies property (H), then so will any other uniform net Y in $\mathcal{H}$.*

Proof. Take $x, y \in \mathcal{H}$ and set, for any $\delta \geqslant 0$,

$$\Upsilon_{(\delta,r)}(x, y) = \{t \in \mathcal{H} \big| xty \text{ is a } \delta - \text{path } d(x, t) \leqslant r\}.$$

Because of property (*H*) we can cover $\Upsilon_{(\delta,r)}(x, y)$ with $P_\epsilon(r)$ balls of radius R_X centered at each point of $\Upsilon_{(\delta,r)}(x, y) \cap X$, where P_ϵ denotes the polynomial associated to the constant $\epsilon = 2R_X + \delta$ in the definition of property (*H*). Now, take $z \in Y$ so that xzy is a δ-path. Since in particular $z \in \mathcal{H}$, we can find $t \in X$ at a distance less than R_X, and thus z is in the ball of radius R_X centered at t. It is now obvious that xty is a $2R_X + \delta$-path. But in each ball of radius R_X there is a uniformly bounded number N of elements of Y, so that

$$\begin{aligned}&\sharp\{t \in Y \text{ such that } d(x, t) \leqslant r, \text{ and } xty \text{ is a } \delta - \text{path}\}\\ &\quad\leqslant N\sharp\{t \in X \text{ such that } d(x, t) \leqslant r, \text{ and } xty \text{ is a } (2R_X + \delta) - \text{path}\}\\ &\quad\leqslant NP_\epsilon(r)\end{aligned}$$

and thus Y satisfies property (*H*), choosing $P_\delta = NP_\epsilon$. □

LEMMA 4.6. *Let $(\mathcal{H}_1, d_1) \dots (\mathcal{H}_n, d_n)$ be metric spaces whose uniform nets all have property (H). Endow $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with the ℓ^1 combination of the distances d_i, then any uniform net in $\mathcal{H}$ has property (H).*

Proof. Because of Lemma 4.5, it is enough to show that one particular uniform net has property (*H*). To do that, let $X_1 \subset \mathcal{H}_1 \dots X_n \subset \mathcal{H}_n$ be uniform nets and look at $X = X_1 \times \cdots \times X_n$, which is a uniform net in $\mathcal{H}$. Take $x, y \in X$, $\delta, r \geqslant 0$ and $t \in \Upsilon_{(\delta,r)}(x, y)$. We write x, y, t in coordinates, that is to say $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $t = (t_1, \dots, t_n)$, where $x_i, y_i, t_i \in X_i$ for any $i = 1, \dots, n$. On each factor X_i, $x_i t_i y_i$ will be a δ-path as well, and since we are considering the ℓ^1 combination of norms, $d_i(x_i, t_i) \leqslant r$. On each X_i there is by assumption at most $P_{\delta,i}(r)$ of those points t_i and thus on X we have at most $P_\delta(r) = \prod_{i=1}^n P_{\delta,i}(r)$ t's at distance to x less than r and such that xty is a δ-path. Obviously P_δ is again a polynomial (of degree the sum of the degrees of the $P_{\delta,i}$'s) and thus X has property (*H*). □

It is an easy observation that any uniform net in a Gromov hyperbolic space has property (*H*), and thus, since any of the metric spaces forming $\mathcal{K}$ have property (*H*), we deduce that $\mathcal{K}$ has property (*H*) as well. Let us now see what happens to property (*K*).

DEFINITION 4.7. A triple $x, y, z \in \mathcal{K}$ forms an *equilateral triangle* if when we write in coordinates $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$, the triples $x_i, y_i, z_i \in \mathcal{X}_i$ form

- an equilateral triangle in the sense given in Definition 3.1 if $\mathcal{X}_i$ is an $X_{\mathbf{K}}$.
- an equilateral triangle in the sense given in [12] if $\mathcal{X}_i$ is an $\widetilde{A}_2$-type building.
- a single point (i.e. $x_i = y_i = z_i$) if $\mathcal{X}_i$ is a hyperbolic space.

Since we endowed $\mathcal{K}$ with the ℓ^1 combinations of the norms on the $\mathcal{X}_i$'s it is obvious that in particular an equilateral triangle will satisfy $d(x, y) = d(y, z) = d(z, x)$. Concerning the orientation, remember that in Definition 3.1 we had two possible orientations for a triangle, positive or negative. In $\mathcal{K}$ we will have much more possible orientations since the orientation of a triangle will depend on its orientation in each non hyperbolic coordinate (in hyperbolic coordinates, the projection is just a point and thus has only one possible orientation). Suppose that among the $\mathcal{X}_i$'s forming $\mathcal{K}$, m of them are not Gromov hyperbolic (for $0 \leqslant m \leqslant n$), and set

$$I = \{(a_1, \ldots, a_m) \big| a_i \in \{+, -\}\},$$

we then say that an equilateral triangle x, y, z has orientation $j \in I$ if in the non hyperbolic components $\mathcal{X}_i$ the triangle x_i, y_i, z_i has the orientation given by a_i for $i = 1, \ldots, m$.

LEMMA 4.8. *For any $\delta_0 > 0$ there exists a $d > 0$ such that the following is true:*

(1) *For any $x, y, z \in \mathcal{K}$ there exists x', y', z' an equilateral triangle in $\mathcal{K}$ such that the paths $xx'y'y$, $yy'z'z$ and $zz'x'x$ are d-paths.*
(2) *For any two equilateral triangles x, y, z and a, b, c in $\mathcal{K}$ of same orientation and such that $d(x, a)$, $d(y, b)$ are both less than δ_0, the triangles x, z, c and y, z, c are d-retractable.*

Proof. (1) For any $i = 1, \ldots, n$ it exists an equilateral triangle x'_i, y'_i, z'_i in $\mathcal{X}_i$ such that the paths $x_i x'_i y'_i y_i$, $y_i y'_i z'_i z_i$ and $z_i z'_i x'_i x_i$ are δ_i-paths. Indeed, this is because of Lemma 3.6 in [9] and Lemma 3.2 in case $\mathcal{X}_i$ is $X_{\mathbf{K}}$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$, because of Section 3 in [12] in case $\mathcal{X}_i$ is an $\tilde{A}_2$-type building and trivially in case $\mathcal{X}_i$ is a hyperbolic space. Now, setting $x' = (x'_1, \ldots, x'_n)$, $y' = (y'_1, \ldots, y'_n)$ and $z' = (z'_i, \ldots, z'_n)$ we see that by construction the triangle x', y', z' is equilateral and that the path $xx'y'y$ is a d-paths for any $d \geqslant \sum_{i=1}^n \delta_i$ since

$$\begin{aligned} d(x, x') &+ d(x', y') + d(y', y) \\ &= \sum_{i=1}^n d_i(x_i, x'_i) + d_i(x'_i, y'_i) + d_i(y'_i, y_i) \\ &\leqslant \sum_{i=1}^n (d_i(x_i, y_i) + \delta_i) \leqslant d(x, y) + d \end{aligned}$$

and similarly for the paths $yy'z'z$ and $zz'x'x$.

(2) Here we have to find a $d \geqslant 0$ and u, v in $\mathcal{K}$ such that the paths xuz, zuc and cux as well as the paths yvz, zvc and cvy are d-paths. But for any $i = 1, \ldots, n$, the triangles x_i, y_i, z_i and a_i, b_i, c_i are equilateral triangles and thus because of Lemma 3.7 in [9] and Lemma 3.2 in case $\mathcal{X}_i$ is $X_{\mathbf{K}}$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$, trivially in case $\mathcal{X}_i$ is an

$\tilde{A}_2$-type building or a hyperbolic space, the triangles x_i, z_i, c_i and y_i, z_i, c_i are δ_i-retractable, that is there exists $\delta_i \geqslant 0$ and points u_i and v_i so that the paths $x_i u_i z_i$, $z_i u_i c_i$ and $c_i u_i x_i$ as well as the paths $y_i v_i z_i$, $z_i v_i c_i$ and $c_i v_i y_i$ are δ_i-paths. Now the points $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are the sought points on which the triangles x, z, c and y, z, c retract, for $d \geqslant \sum_{i=1}^n \delta_i$. Indeed, let us check that for the path xuz:

$$\begin{aligned} &d(x, u) + d(u, z) \\ &\quad = \sum_{i=1}^{n} (d_i(x_i, u_i) + d(u_i, z_i)) \\ &\quad \leqslant \sum_{i=1}^{n} (d_i(x_i, z_i) + \delta_i) \leqslant d(x, z) + d. \end{aligned}$$

and similarly for the paths zuc and cux as well as the paths yvz, zvc and cvy. □

The following lemma is the analogue of a part of Theorem 3.3 in [9].

LEMMA 4.9. *Let Γ be a discrete cocompact subgroup of the isometry group of $\mathcal{K}$, and $Z \subset \mathcal{K}$ be a Γ-invariant uniform net. Let X be a free Γ-space and $\theta\colon X \to Z$ be a Γ-equivariant map. Endow X with the distance*

$$\theta^*(d)(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1 + d(\theta(x), \theta(y)), & \text{if } x \neq y, \end{cases}$$

where d is the induced distance of $\mathcal{K}$ on Z. Then the pair (X, Γ) satisfies property (K).

Proof. Take $\delta \geqslant 4R_Z + d$ (for d as in Lemma 4.8). We first have to define the Γ-invariant subsets of X^3. Let us consider I as explained in Definition 4.7, that is, I is a set of indices running along the possible orientations of equilateral triangles in $\mathcal{K}$. For any $i \in I$, we define

$$\mathcal{T}_i' = \{(\alpha, \beta, \gamma) \in Z^3 \mid \text{ it exists } (a, b, c) \in \mathcal{K}^3 \text{ equilateral triangle with } d(\alpha, a) \leqslant R_Z, d(\beta, b) \leqslant R_Z, d(\gamma, c) \leqslant R_Z\}$$

In other words, $\mathcal{T}_i'$ is the set of triples of Z which are not too far from an equilateral triangle, and since Γ acts on $\mathcal{K}$ by isometries, the sets $\mathcal{T}_i'$ are Γ-invariant. We then set, for any $i \in I$, $\mathcal{T}_i = \theta^{-1}(\mathcal{T}_i')$. Since θ is Γ-equivariant, $\mathcal{T}_i$ is Γ-invariant for any $i \in I$. Let us explain why then (X, Γ) satisfy property (K). Because of the distance defined on X, it is enough to prove (K_a) and (K_b) for Z and the sets $\mathcal{T}_i'$.

<u>(K_a)</u>: Take $(x, y, z) \in Z^3$, we have to show that there exists (α, β, γ) in some $\mathcal{T}_i$ so that the triple (x, y, z) retracts on (α, β, γ). But because of part a) of Lemma 4.8, we know that there exists $(x', y', z') \in \mathcal{K}^3$, forming an equilateral triangle and so that the triple x, y, z retracts on x', y', z'. Now Z being a uniform net in $\mathcal{K}$, there exists α, β, γ three points of Z with $d(\alpha, x') \leqslant R_Z$, $d(\beta, y') \leqslant R_Z$ and $d(\gamma, z') \leqslant R_Z$, so that the triple (α, β, γ) belongs to $\mathcal{T}_i'$ for some $i \in I$. We compute:

$$\begin{aligned}
&d(x,\alpha)+d(\alpha,\beta)+d(\beta,y)\\
&\quad\leqslant d(x,x')+d(x',\alpha)+d(\alpha,x')+d(x',y')+d(y'\beta)+d(\beta,y')+d(y',y)\\
&\quad\leqslant d(x,x')+d(x',y')+d(y',y)+4R_Z\\
&\quad\leqslant d(x,y)+d+4R_Z\leqslant\delta
\end{aligned}$$

and similarly for the paths $y\beta\gamma z$ and $z\gamma\alpha x$.

$\underline{(K_b)}$: Take $i\in I$ and four points in Z defining two triples in $\mathcal{T}'_i$, say (α,β,γ) and (α,β,γ'). We have to prove that both triangles α,γ,γ' and β,γ,γ' are δ-retractable. By definition of $\mathcal{T}'_i$ we can find two equilateral triangles a,b,c and x,y,z such that

$$d(\alpha,a)\leqslant R_Z,\ d(\beta,b)\leqslant R_Z,\ d(\gamma',c)\leqslant R_Z$$

and

$$d(\alpha,x)\leqslant R_Z,\ d(\beta,y)\leqslant R_Z,\ d(\gamma,z)\leqslant R_Z$$

so that obviously $d(a,x)\leqslant 2R_Z$ and $d(b,y)\leqslant 2R_Z$ and thus applying part b) of Lemma 4.8 we have the existence of a $d\geqslant 0$, and two points u and v in $\mathcal{K}$ so that the paths xuz, zuc and cux as well as the paths yvz, zvc and cvy are d-paths. Again, Z being a uniform net in $\mathcal{K}$, we can find u' and v' in Z at respective distances less than R_Z to u and v. We claim that the paths $\alpha u'\gamma$, $\gamma u'\gamma'$ and $\gamma' u'\alpha$ as well as the paths $\beta v'\gamma$, $\gamma v'\gamma'$ and $\gamma' v'\beta$ are δ-paths:

$$\begin{aligned}
&d(\alpha,u')+d(u',\gamma)\\
&\quad\leqslant d(\alpha,x)+d(x,u)+d(u,u')+d(u',u)+d(u,z)+d(z,\gamma)\\
&\quad\leqslant 4R_Z+d(x,u)+d(u,z)\leqslant d(x,z)+d+4R_Z\\
&\quad\leqslant d(x,\alpha)+d(\alpha,\gamma)+d(\gamma,z)+d+4R_Z\\
&\quad\leqslant d(\alpha,\gamma)+(d+6R_Z)\leqslant d(\alpha,\gamma)+\delta
\end{aligned}$$

and similarly for the other paths. □

Now, if under the assumptions of this lemma we furthermore assume that $\sharp\theta^{-1}(z)\leqslant N$ (i. e. θ has uniformly bounded fibers) we use Remark 4.2 and Lemma 4.5 to deduce that X has property (H). We now can apply Theorem 4.4, as follows: Z is a Γ-invariant uniform net in $\mathcal{K}$ and let $Z=\coprod_{j\in J}\Gamma x_j$ its partition in Γ-orbits. Then with $X=\coprod_{i\in J}\Gamma$ and θ the obvious orbit map and with $\{\mathcal{T}_i\}_{i\in I}$ as defined in the previous lemma we get:

THEOREM 4.10. *Let Γ be a discrete group acting by isometries on $\mathcal{K}$ and with uniformly bounded stabilizers on some Γ-invariant uniform net. Then Γ has the property (RD).*

If Γ is a cocompact lattice of isometries on $\mathcal{K}$, it is enough to take $Z=\Gamma x_0$ and $X=\Gamma$, so Γ has the property (RD).

Remark 4.11. If one is only interested in cocompact lattices in the isometry group of a product of proper hyperbolic spaces the proof of the property (RD) then becomes much simpler. Indeed, denote by $\mathcal{H}$ a finite product of hyperbolic spaces, Γ a cocompact lattice in its isometry group and $X = \Gamma x_0$ for some $x_0 \in \mathcal{H}$. We showed that X has the property (H), so it remains to show that any triple of points is δ-retractable, but this is a special case of Lemma 4.8 part 1). We can conclude using Proposition 2.3 of [9]. This is done explicitely in [1], and following the advice of N. Higson, the case of a finite product of nonlocally finite trees is treated as well, yielding property (RD) for Coxeter groups.

Notice that for this work we have established property (RD) for Γ a cocompact lattice of isometries on $\mathcal{K}$, endowed with the ℓ^1 combination of the distances, and this implies Theorem 0.1 since $\mathrm{Iso}(\mathcal{X}_1) \times \cdots \times \mathrm{Iso}(\mathcal{X}_n) \subset \mathrm{Iso}(\mathcal{K}, \ell^1)$. The arguments used for the proof fail for other combinations of the distances.

While this article was being referred, we learnt that independently M. Talbi in [15] established property (RD) for cocompact lattices in the isometry group of an arbitrary finite product of type $\tilde{A}_{i_1} \times \cdots \times \tilde{A}_{i_n}$ for $i_j = 1, 2$, which is a special case of Theorem 4.10.

As already noticed by V. Lafforgue, these methods (and more specifically points (1) and (2) of Lemma 4.8) fail to be true beyond the $A_{i_1} \times \cdots \times A_{i_k} \times \tilde{A}_{i_{k+1}} \times \cdots \times \tilde{A}_{i_n}$ cases, for $i_j = 1, 2$, and thus new ideas are needed to establish property (RD) for cocompact lattices in $Sp_4(\mathbf{R})$ or $SL_4(\mathbf{R})$.

Finally, it has been pointed out by the referee that it would be interesting to know whether discrete subgroups of the Lie groups of Theorem 0.1 have property (RD) if they do not contain unipotent elements, but unfortunately these methods don't apply, as cocompactness is crucial to deduce Lemma 4.9 out of Lemma 4.8.

Acknowledgements

It is a pleasure to thank my thesis advisor A. Valette for suggesting to me this problem and for all his generous help. I must also thank V. Lafforgue for graciously explaining the methods of [12] and [9]. I am also extremely indebted to D. Allcock and G. Prasad for important discussions on the exceptional group $E_{6(-26)}$.

References

1. Chatterji, I.: Property (RD) for certain discrete groups, PhD Thesis, ETH Zürich, 2001.
2. Ferry, S., Ranicki, A. and Rosenberg, J.: *Novikov Conjectures, Index Theorems and Rigidity*, London Math. Soc. Lecture Note Ser. 226, Cambridge Univ. Press, 1993.
3. Freudenthal, H.: Oktaven, Ausnahmegruppen und Oktavengeometrie, *Geom. Dedicata* **19** (1985), 1–73.
4. Freudenthal, H.: Zur Ebenen Oktavengeometrie, *Neder. Akad. van Wetenschappen* **56** (1953), 195–200.
5. Haagerup, U.: An example of nonnuclear C^*-algebra which has the metric approximation property, *Invent. Math.* **50** (1979), 279–293.
6. de la Harpe, P.: Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint, *C.R. Acad. Sci. Paris Sér. I* **307** (1988), 771–774.

7. Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Pure Appl. Math. 80, Academic Press, New York, 1978.
8. Jolissaint, P.: Rapidly decreasing functions in reduced C^*-algebras of groups, *Trans. Amer. Math. Soc.* **317** (1990), 167–196.
9. Lafforgue, V.: A proof of property (RD) for discrete cocompact subgroups of $\mathrm{SL}_3(\mathbf{R})$, *J. Lie Theory* **10** (2000), 55–277.
10. Lafforgue, V.: Une démonstration de la conjecture de Baum–Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T), *C.R. Acad. Sci. Paris Ser. I Math.* **327** (5) (1998), 439–444.
11. Lubotzky, A., Mozes, S. and Raghunathan, M. S.: The word and Riemannian metrics on lattices of semisimple groups, *Inst. Hautes Etudes Sci. Publ. Math.* **91** (2000), 5–53.
12. Ramagge, J., Robertson, G. and Steger, T.: A Haagerup inequality for $\tilde{A}_1 \times \tilde{A}_1$ and $\tilde{A}_2$ buildings, *Geom. Funct Anat.* **8** (1988), 702–731.
13. Schafer, R. D.: *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.
14. Springer, T.: *Octonions, Jordan Algebras and Exceptional Groups*, Springer Verlag, New York.
15. Talbi, M.: Inégalité de Haagerup et géométrie des groupes, PhD Thesis, Université Lyon I (France), 2001.
16. Valette, A.: *An Introduction to the Baum–Connes Conjecture*, From notes taken by I. Chatterji, with an appendix by G. Mislin, Lectures Notes in Math., ETH Zürich, Birkhäuser-Verlag, Basel, 2002.