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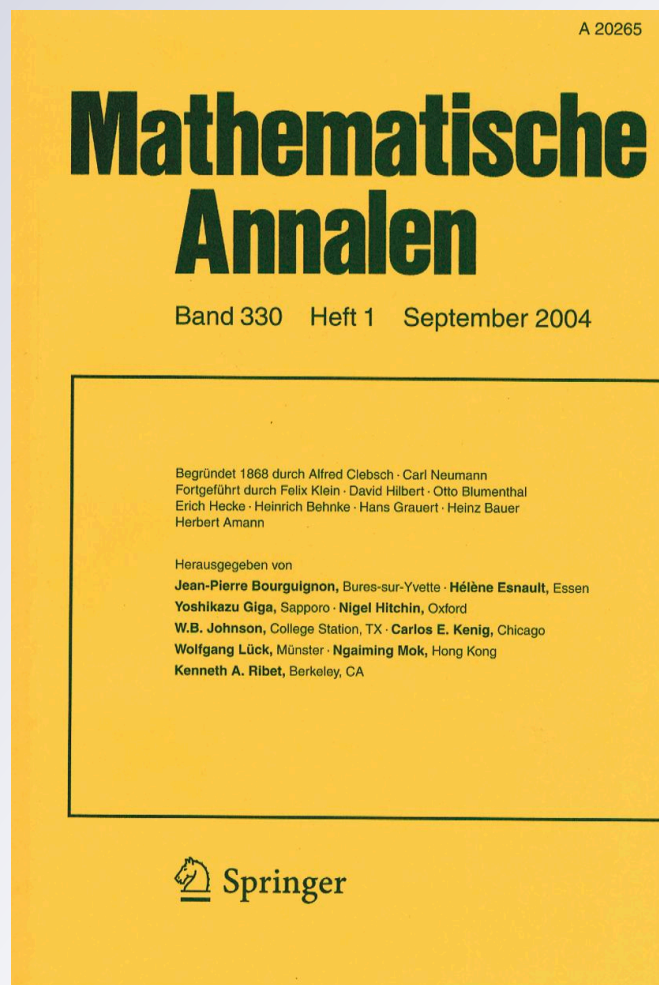
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A geometric criterion for the boundedness of characteristic classes

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Abstract We show that for a connected Lie group G , the linearity of its radical \sqrt{G} (that is of its biggest connected normal solvable subgroup), is a necessary and sufficient condition for the boundedness of all Borel cohomology classes of G with integer coefficients, and that the linearity of \sqrt{G} is also equivalent to a large-scale geometric property of G (involving distortion).

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1 Introduction

Let G be a Lie group and let G^δ be the same group endowed with the discrete topology. The identity map $G^\delta \rightarrow G$ induces a map between classifying spaces $BG^\delta \rightarrow BG$. Gromov proved in 1982 that if G is the real Lie group associated to a linear algebraic group defined over \mathbb{R} , then the range of the natural map

$$H^*(BG, \mathbb{R}) \rightarrow H^*(BG^\delta, \mathbb{R}),$$

between the singular cohomology groups, consists of bounded classes. In other words, following the terminology introduced in [17, Sect. 1.3, p. 23]: *primary characteristic classes of a linear algebraic group over \mathbb{R} are bounded*. In view of a result proved by Wigner in 1973 [36, Theorem 4, p. 93], Gromov’s result may be reformulated as follow: *all Borel cohomology classes with integer coefficients of a linear algebraic group over \mathbb{R} are bounded*.

It turns out that *neither the algebraicity nor the linearity of the group are necessary conditions in Gromov’s result*: relying on Gromov’s result and its proof by Bucher-Karllsson, we show in Sect. 5 that *the linearity of the radical of G is a sufficient condition for the boundedness of all primary characteristic classes of G* . Recall that there are connected Lie groups which are linear but not algebraic. Also there are connected Lie groups with linear radical but which do not cover any linear Lie group. (The quotient of the product of the three dimensional Heisenberg group with the universal cover of $SL(2, \mathbb{R})$ by a central diagonal infinite cyclic subgroup is such an example.) The linearity of the radical of a connected Lie group G is still too strong an hypothesis for the boundedness of all the primary characteristic classes of G , see Remark 55 at the end of the paper.

Our main result is that *the linearity of the radical of a connected Lie group G , is a necessary and sufficient condition for the boundedness of all Borel cohomology classes of G with integer coefficients, and that the linearity of the radical is also equivalent to a large-scale geometric property of G (involving distortion)*.

We recall the definitions needed to state our results. A Borel n -cochain on G with values in \mathbb{Z} is a map

$$c : G^n \rightarrow \mathbb{Z},$$

such that the inverse image of any subset of \mathbb{Z} is a Borel set (i.e. an element of the σ -algebra generated by the open subsets of G^n endowed with its product topology). The usual differentials send Borel cochains to Borel coboundaries. We denote by

$$H_B^*(G, \mathbb{Z})$$

the corresponding cohomology group (see Sect. 3 below for details and references).

Recall that a class in $H_B^*(G, \mathbb{Z})$ is *bounded* if it admits a representative Borel cocycle whose image is finite.

We refer the reader to Subsect. 2.2, Definition 15 below for the definition of an *undistorted* subgroup.

Theorem 1 *Let G be a connected Lie group. The following conditions are equivalent.*

1. *The radical of G is linear.*
2. *Each Borel cohomology class of G with \mathbb{Z} -coefficients can be represented by a Borel bounded cocycle.*
3. *The natural inclusion $\pi_1(G) \rightarrow \tilde{G}$ of the fundamental group of G into the universal cover of G is undistorted.*

The implication (1) \Rightarrow (2) is proved in two steps. We first show that (1) implies that 2-dimensional integral Borel cohomology classes are bounded, using an interpretation of central extensions of topological groups in terms of Borel cocycles, due to Moore [30]. We then show, relying on the boundedness result on real primary characteristic classes of flat bundles with structure group a real linear algebraic Lie group, due to Gromov [17, Sect. 1.3, p. 23] and Bucher-Karlsson [9], how to pass to cohomology classes of degree larger than 2. The implication (2) \Rightarrow (3) relies on properties of distorted discrete central subgroups of Lie groups and the equivalence of (1) and (3) is dealt with using Lie group and Lie algebra techniques.

In Sect. 2 we recall some facts on Lie groups and prove the results concerning distortion of subgroups used later on. Borel cohomology of topological groups, which is based on Borel (measurable) cochains, is discussed in Sect. 3. In Sect. 4 we present the proof of Theorem 1 and mention in Corollary 53 two more equivalent conditions to the ones of Theorem 1. In Sect. 5 we use Wigner's result mentioned above to explain the relationship of Theorem 1 with Gromov's Theorem.

Theorem 1 has roots in [11, Proposition 5.5 and Lemma 6.3], where some of the authors of the present paper needed to work with Borel cocycles associated to distorted and undistorted central extensions.

2 On the geometry and topology of Lie groups

2.1 Preliminaries and facts on Lie groups

In this subsection, we discuss general facts about the topology of Lie groups and their algebraic structure. We also recall linearity criteria for Lie groups (Theorem 2 and Proposition 10).

To fix notation, we recall some basic facts on Lie groups; a good reference is [32]. Let G be a connected (real) Lie group. Then G admits a Levi decomposition $G = \sqrt{G} \cdot L(G)$ with \sqrt{G} the radical of G (the biggest connected solvable normal subgroup) and $L(G)$ a Levi subgroup (a maximal connected semi-simple subgroup; the subgroups \sqrt{G} and $L(G)$ are analytic (a subgroup $H < G$ is called *analytic*, if H is a connected Lie group and the inclusion $H \rightarrow G$ is an immersion of manifolds). The intersection $\sqrt{G} \cap L(G)$ is a totally disconnected subgroup of G , which is discrete in $L(G)$ but in general not discrete in \sqrt{G} . In case G is simply-connected, $\sqrt{G} \cap L(G) = \{e\}$; hence G is the semi-direct product $G = \sqrt{G} \rtimes L(G)$. A group G is called *linear*, if it admits a faithful representation $G \rightarrow GL(n, \mathbb{R})$. In case G is a connected Lie group, there is a closed normal subgroup $\Lambda(G)$, the *linearizer* of G , such that $G/\Lambda(G)$ is linear and such that any Lie homomorphism $G \rightarrow H$ with

H linear factors through $G/\Lambda(G)$ (for the structure of the linearizer see [23]). If G is connected and semi-simple, $\Lambda(G)$ is a central discrete subgroup and any quotient Lie group of $G/\Lambda(G)$ is linear too [32, Chap. 5, Sect. 3, Theorem 8]. In case G is connected and solvable, $\Lambda(G)$ is a central torus; a quotient of a linear solvable Lie group need not be linear. Concerning the linearity of Lie groups, the following theorem is basic.

Theorem 2 (Malcev [26].) *A connected Lie group G is linear if and only if its radical \sqrt{G} and Levi subgroup $L(G)$ are.*

In the sequel, we will also deal with not necessarily connected Lie groups G ; we will write G^0 for the connected component of G . By the radical \sqrt{G} of G we still mean a maximal connected normal solvable subgroup of G , thus $\sqrt{G} = \sqrt{G^0}$. A group G is called *virtually connected*, if G/G^0 is a finite group. In case G_a is a connected linear algebraic group defined over \mathbb{R} , its Lie group of \mathbb{R} -points $G_a(\mathbb{R})$ is virtually connected. Every connected linear reductive Lie group G is, as a Lie group, isomorphic to $G_a(\mathbb{R})^0$ for some connected linear real algebraic group G_a (see [25]).

The following simple observation is used later.

Lemma 3 *Let G be a contractible Lie group and $H < G$ an analytic subgroup. Then H is contractible too.*

Proof Recall that any connected Lie group is homotopy equivalent to a maximal compact subgroup. Let K be a maximal compact subgroup of H . Then K is contained in a maximal compact subgroup of G , which is $\{e\}$. Thus $K = \{e\}$ and therefore H is contractible.

Since a maximal compact subgroup of a connected solvable Lie group S is a torus, S is contractible if and only if S is simply-connected. It follows that every connected analytic subgroup of a simply-connected solvable Lie group is contractible.

Lemma 4 *Let N be a connected closed normal subgroup of a connected Lie group G . Then the inclusion $N \hookrightarrow G$ induces a short exact sequence $0 \rightarrow \pi_1(N) \rightarrow \pi_1(G) \rightarrow \pi_1(G/N) \rightarrow 0$.*

Proof The long exact sequence in homotopy reads

$$\dots \rightarrow \pi_2(G/N) \rightarrow \pi_1(N) \rightarrow \pi_1(G) \rightarrow \pi_1(G/N) \rightarrow 0.$$

Because G/N is a Lie group, $\pi_2(G/N) = 0$ (see [7]), and the result follows.

Remark 5 Since \mathbb{R} is divisible, $\text{Hom}(-, \mathbb{R})$ is an exact contravariant functor on abelian groups. As the fundamental group of a Lie group is abelian, Lemma 4 shows that the inclusion $N \hookrightarrow G$ induces a surjection

$$\text{Hom}(\pi_1(G), \mathbb{R}) \twoheadrightarrow \text{Hom}(\pi_1(N), \mathbb{R}).$$

Lemma 6 *Let N be a closed connected normal subgroup of a connected Lie group G . There are two exact sequences with commutative squares*

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \Gamma & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & \Gamma \cap \tilde{N} & \longrightarrow & \tilde{N} & \longrightarrow & N & \longrightarrow & 1,
 \end{array}$$

where \tilde{G} and \tilde{N} are the respective universal covers of G and N , whereas Γ and $\Gamma \cap \tilde{N}$ are central discrete subgroups, isomorphic to the respective fundamental groups of G and N , and the vertical arrows are injections of closed normal subgroups.

Proof Let $p : \tilde{G} \rightarrow G$ be the universal cover of G . The connected component C of the identity in $p^{-1}(N)$ is a closed connected normal subgroup of \tilde{G} and the restriction of p to C is a cover of N . According to Lemma 4 the group C is simply-connected. Hence $C = \tilde{N}$, the universal cover of N .

Lemma 7 *Let G be a connected nilpotent Lie group. Let $\phi : \mathbb{R} \rightarrow G$ be a one-parameter subgroup. If $\phi(t_0)$ is central for some $t_0 \neq 0$ then $\phi(t)$ is central for all t .*

Proof As the universal cover of a Lie group G is an extension of G with discrete central kernel, an easy argument shows that it is enough to prove the lemma when the nilpotent group G is simply-connected. In this case, the exponential map

$$\exp : \mathfrak{g} \rightarrow G$$

is a diffeomorphism. In exponential coordinates the Lie group multiplication is given by the Campbell-Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \dots$$

for all $x, y \in \mathfrak{g}$. As the Lie algebra \mathfrak{g} is nilpotent, the above expression has finitely many terms. Hence if $g \in G$ is given, the equation $g\phi(t)g^{-1}\phi(t)^{-1} = e$ is given by $d = \dim(\mathfrak{g})$ polynomial equations $P_i(t) = 0, 1 \leq i \leq d$. By hypothesis, for each i , we have $P_i(nt_0) = 0$ for all integer n . Hence the equality holds for all $t \in \mathbb{R}$.

The following well-known fact is an immediate consequence.

Lemma 8 *A connected nilpotent Lie group N has a central torus as its unique maximal compact subgroup.*

Proof Since a connected compact subgroup of N is a torus, and since the union of all 1-dimensional tori in a given torus T is a dense subset of T , it suffices to prove that every 1-dimensional torus $S^1 < N$ is central. Choose a surjective homomorphism $\phi : \mathbb{R} \rightarrow S^1 < N$ and $t_0 \in \mathbb{R} \setminus \{0\}$ with $\phi(t_0) = e$. Then the results follows by applying Lemma 7.

Recall that the *nilradical* of a Lie group is the largest connected, normal, nilpotent subgroup.

Lemma 9 *Let K be a compact subgroup of a connected solvable Lie group G . Let T_N be the maximal compact subgroup of the nilradical N of G . If $k \in [G, G] \cap K$ then there exists a one-parameter subgroup in $T_N \cap [G, G]$ passing through k .*

Proof Let $k \in [G, G] \cap K$ be a non-trivial element. Let \mathfrak{D} be the Lie algebra of the (not necessarily closed) analytic subgroup $[G, G]$. Since G is solvable, $[G, G]$ is nilpotent and the exponential map

$$\exp : \mathfrak{D} \rightarrow [G, G]$$

is surjective (see [22]). Let $X \in \mathfrak{D}$ be such that $\exp(X) = k$ and denote by $\phi : \mathbb{R} \rightarrow [G, G]$ the corresponding one-parameter subgroup $\phi(t) = \exp(tX)$. Since $[G, G]$ is a normal nilpotent connected subgroup of G , it is contained in the nilradical N of G . Let T_N be the maximal compact subgroup of N . We hence have that $[G, G] \cap K \subseteq T_N$. Since $k = \exp(X) \in T_N$, it implies that $\phi(\mathbb{Z}) \subseteq T_N$. The 1-parameter subgroup $\phi(\mathbb{R})$, being at bounded distance from $\phi(\mathbb{Z})$ (for any left-invariant Riemannian metric on G), is bounded and hence the closure of $\phi(\mathbb{R})$ in N is a compact subgroup. We conclude that $\phi(\mathbb{R}) \subseteq T_N \cap [G, G]$.

Proposition 10 (Malcev, Gotô) *Let G be a connected solvable Lie group. The following are equivalent:*

1. *The group G is linear.*
2. *The group $\pi_1([G, G])$ is trivial.*
3. *There is a maximal compact subgroup $K < G$ and a closed 1-connected normal subgroup $H < G$ such that $G = HK$.*

The equivalence of (1) and (2) was proved by Gotô, see [16, Theorem 5], and that of (1) and (3) is due to Malcev [26].

Remark 11 In case the group G is nilpotent, the equivalent conditions of Proposition 10 are also equivalent to the following one: *the group is a direct product $G = T \times N$ where T is the unique maximal compact subgroup of G and where N is nilpotent and contractible.* This follows from Lemma 8.

Remark 12 The equivalent conditions of Proposition 10 are not equivalent to the group $\pi_1([G, G])$ being trivial. Indeed, the following is an example of a connected solvable Lie group G with $[G, G]$ simply-connected but $\overline{[G, G]}$ not. Let \mathbf{H} be the 3-dimensional Heisenberg group and consider $\mathbf{H} \times S^1$. Its center is $\mathbb{R} \times S^1$. Take the discrete central subgroup \mathbb{Z} generated by $(1, t)$ with 1 generating \mathbb{Z} in \mathbb{R} , and t of infinite order in S^1 ; this central subgroup of $\mathbf{H} \times S^1$ is discrete. Let us define $G := (\mathbf{H} \times S^1)/\mathbb{Z}$. It is a nilpotent connected Lie group with $[G, G]$ homeomorphic to \mathbb{R} , embedded in the maximal torus $S^1 \times S^1$ of G in a dense way. It follows that $\pi_1([G, G])$ is trivial but $\pi_1(\overline{[G, G]}) = \mathbb{Z} \times \mathbb{Z}$. Thus G is not linear (its linearizer is $\overline{[G, G]}$, a central 2-torus). We would like to warn the reader that in several places in the literature one finds an

incorrect statement saying that a connected solvable Lie group G is linear if and only if $[G, G]$ is simply-connected (for instance, see [35, Chap. 2, Theorem 7.1]); the correct statement is that G is linear if and only if the closure $\overline{[G, G]} < G$ is simply-connected.

For our purpose the following variation of Proposition 10 will be useful.

Lemma 13 *Let G be a connected solvable Lie group. The following are equivalent:*

1. *The group G is linear.*
2. *For every maximal compact subgroup $K < G$, $\overline{[G, G]} \cap K = \{e\}$.*
3. *For every maximal compact subgroup $K < G$, $[G, G] \cap K = \{e\}$.*

Proof (1) \Rightarrow (2) \Rightarrow (3): If G is linear, $G = HK$ with $K < G$ maximal compact and H simply-connected, normal and closed (cf. Proposition 10); we can choose here any maximal compact subgroup of $K < G$, because these are all conjugate. This implies that $H \cap K = \{e\}$ so that $G/H \cong K$ is commutative, because K is a torus. Thus $\overline{[G, G]} < H$ and (2) and then (3) follow. It remains to show that (3) implies (1). If G is not linear, the maximal compact subgroup T of $\overline{[G, G]}$ is a non-trivial torus because $\pi_1(\overline{[G, G]}) \neq \{e\}$. The assumption (3) implies that $T \cap [G, G] = \{e\}$. Because $\overline{[G, G]}$ is nilpotent, T is a central subgroup of $\overline{[G, G]}$ (Lemma 8). Therefore,

$$\phi : T \times [G, G] \rightarrow \overline{[G, G]}, \quad (t, x) \mapsto tx$$

maps isomorphically onto the dense subgroup $T[G, G] < \overline{[G, G]}$. Under the projection $\overline{[G, G]} \rightarrow \overline{[G, G]}/T$, $[G, G]$ maps isomorphically onto a dense analytic subgroup $U < \overline{[G, G]}/T$. Since $\overline{[G, G]}/T$ is simply-connected and U is normal, U is closed and therefore equal to $\overline{[G, G]}/T$. It follows that the pre-image $T[G, G]$ is equal to $\overline{[G, G]}$ and we conclude that ϕ is an isomorphism of Lie groups. This implies that $[G, G]$ is dense in $T \times [G, G]$, which contradicts the assumption that T is non-trivial.

Lemma 14 *Let G be a connected Lie group, \sqrt{G} its radical and $Q = G/\sqrt{G}$ simply-connected. The short exact sequence*

$$1 \rightarrow \sqrt{G} \rightarrow G \rightarrow Q \rightarrow 1$$

splits, and therefore we have a semi-direct product, $G = \sqrt{G} \rtimes Q$.

Proof This is a classical result in the case G itself is simply-connected (see [22]). The proof in our case is reduced to the simply-connected case by considering the following two exact sequences with commutative squares,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \sqrt{\tilde{G}} = p^{-1}(\sqrt{G})^0 & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{G}/\sqrt{\tilde{G}} \longrightarrow 1 \\ & & \downarrow & & \downarrow p & & \downarrow \phi \\ 1 & \longrightarrow & \sqrt{G} & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1, \end{array}$$

where \tilde{G} is the universal cover of G . The map ϕ is an isomorphism because it is a connected covering of a simply-connected space. Thus, if σ is a splitting for $\tilde{G} \rightarrow \tilde{G}/\sqrt{\tilde{G}}$, then $p \circ \sigma \circ \phi^{-1}$ is a splitting for $G \rightarrow Q$.

2.2 Distorted and undistorted central extensions

In this subsection, we first recall some basic facts about the distortion of subgroups. We establish an algebraic criterion to decide when certain central subgroups of a simply-connected solvable Lie group are distorted (Proposition 19). Finally, we show that if the radical of a connected Lie group G is not linear, then the fundamental group of G is distorted in the universal cover of G (Proposition 22).

Definition 15 Let A and E be two locally compact, compactly generated groups. We denote by L_S , resp. L_U , the word length associated to a symmetric relatively compact generating set S of A , resp. U of E . Assume that A is a subgroup of E . We say that A is *undistorted* in E if the identity map is a quasi-isometry between (A, L_S) and $(A, L_U|_A)$. Otherwise we say that the subgroup A is *distorted* in E . If A is the kernel of a continuous homomorphism $E \rightarrow G$, we call the extension

$$1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$$

undistorted, resp. *distorted*, if A is undistorted, resp. distorted in E .

Remarks 16 (1) Under the hypothesis of the above definition, Gromov defines in [18, Chap. 3], the distortion function as

$$\text{DISTO}(r) := \frac{\text{diam}_A(A \cap B_E(r))}{r}, \quad \forall r > 0.$$

One checks that A is undistorted exactly when the function DISTO is bounded.

(2) We say that the distortion of A is *at least linear* if there exist $a > 0$ and $R > 0$ such that for all $r \geq R$, we have

$$ar \leq \text{DISTO}(r).$$

We say that the distortion is *sub-linear* if it is not at least linear.

(3) Having a bounded, or unbounded, or sub-linear, etc., distortion function, is a well defined property of the couple $A < E$, i.e. does not depend on the choice of the relatively compact symmetric generating sets.

Lemma 17 Let $p : G \rightarrow Q$ be a continuous homomorphism between locally compact, compactly generated groups. Let $H < G$ be a compactly generated subgroup of G and assume that $H \cap \ker(p) = \{e\}$. If H is distorted in G then so is $p(H)$ in Q .

Proof Let S be a compact symmetric generating set of H . Then $p(S)$ is a compact symmetric generating set of $p(H)$. Let $h \in H$. As p is a homomorphism, $L_{p(S)}(p(h)) \leq L_S(h)$. As $H \cap \ker(p) = \{e\}$, we also have $L_{p(S)}(p(h)) \geq L_S(h)$. The proof now follows because a homomorphism is C -Lipschitz with respect to word metrics (for $C \geq 1$ a constant depending on p and on generating sets) and because the projection of a path between the identity in G and $h \in H$ is still a path between the identity in Q and $p(h) \in p(H)$.

Proposition 18 (Gromov [18, 3B2].) *Let N be a simply-connected nilpotent Lie group with Lie algebra \mathfrak{n} . Let $x \in [\mathfrak{n}, \mathfrak{n}] \setminus \{0\}$. Then the one-parameter subgroup $t \mapsto \exp(tx)$, is at least linearly distorted in N .*

The statement in [18, 3B2] which implies Proposition 18 is not proved. For a proof of Proposition 18, we refer the reader to [34, Prop. 4.1] or [33]. A more conceptual proof, in the spirit of [18], follows from the existence of a homothety, relative to a Carnot-Carathéodory metric on N , in the case N is graded. The general case is reduced to the graded case using [6, Theorem 1.3].

Proposition 19 *Let $1 \rightarrow N \rightarrow S \rightarrow A \rightarrow 1$ be a short exact sequence of connected Lie groups with N nilpotent S simply-connected and A abelian. Let $Z < N$ be a connected analytic subgroup, which is central in S . Denote by \mathfrak{z} , resp. \mathfrak{s} the Lie algebras of Z , resp. S . Then either \mathfrak{z} is a direct factor in \mathfrak{s} , or there exists a one-parameter subgroup in Z which is distorted in S .*

Proof Let $T \in \mathfrak{s}$ be a regular element, and let $\mathfrak{n}(T, \mathfrak{s})$ be the associated Cartan sub-algebra [12, Chapitre VI, Paragraphe 4, 2, Proposition 9, p. 387]. The sub-algebra $\mathfrak{n}(T, \mathfrak{s})$ is the big kernel of $ad(T)$; that is, there is an integer $N \in \mathbb{N}$, big enough such that $\mathfrak{n}(T, \mathfrak{s}) = \ker(ad^N(T)) = \ker(ad^{N+1}(T))$. Let $\mathfrak{i}(T, \mathfrak{s})$ be the small image of $ad(T)$; that is, there is an integer $M \in \mathbb{N}$, big enough such that $\mathfrak{i}(T, \mathfrak{s}) = ad^M(T)(\mathfrak{s}) = ad^{M+1}(T)(\mathfrak{s})$. Hence \mathfrak{s} decomposes as a direct sum of $ad(T)$ -invariant subspaces,

$$\mathfrak{s} = \mathfrak{n}(T, \mathfrak{s}) \oplus \mathfrak{i}(T, \mathfrak{s}).$$

The derivation $ad(T)$ has a semi-simple part $ad_s(T)$, that gives $\mathfrak{n}(T, \mathfrak{s}) = \ker(ad_s(T))$, and $\mathfrak{i}(T, \mathfrak{s}) = ad_s(T)(\mathfrak{s})$, see [12, Chapitre VI, Paragraphe 4, 2, Proposition 8, p. 385]. It follows that

$$[\mathfrak{n}(T, \mathfrak{s}), \mathfrak{i}(T, \mathfrak{s})] \subseteq \mathfrak{i}(T, \mathfrak{s}),$$

because if $X \in \mathfrak{n}(T, \mathfrak{s})$ and $Y \in \mathfrak{i}(T, \mathfrak{s})$, we can choose $\tilde{Y} \in \mathfrak{s}$ such that $ad(T)(\tilde{Y}) = Y$, hence

$$[X, Y] = [X, ad_s(T)(\tilde{Y})] + [ad_s(T)(X), \tilde{Y}] = ad_s(T)([X, \tilde{Y}]).$$

We note that the Cartan subgroup $C < S$ associated to $\mathfrak{n}(T, \mathfrak{s})$ is, by Lemma 3, a simply-connected nilpotent group, because S is a simply-connected solvable Lie group and therefore contractible. If the intersection $\mathfrak{z} \cap [\mathfrak{n}(T, \mathfrak{s}), \mathfrak{n}(T, \mathfrak{s})]$ is non-trivial, then according to Proposition 18, there is a one-parameter subgroup in Z which is distorted in the Cartan subgroup $C < S$. This obviously implies that Z is distorted in S as well, and the proof is finished in this case.

So we can assume that $\mathfrak{z} \cap [\mathfrak{n}(T, \mathfrak{s}), \mathfrak{n}(T, \mathfrak{s})] = \{0\}$. We notice that as the sub-algebra \mathfrak{z} is central in \mathfrak{s} , it is contained in $\mathfrak{n}(T, \mathfrak{s})$. Hence we can choose a complement V for $\mathfrak{z} \oplus [\mathfrak{n}(T, \mathfrak{s}), \mathfrak{n}(T, \mathfrak{s})]$ in $\mathfrak{n}(T, \mathfrak{s})$. Obviously $\mathfrak{m} = [\mathfrak{n}(T, \mathfrak{s}), \mathfrak{n}(T, \mathfrak{s})] \oplus V$ is a sub-algebra of $\mathfrak{n}(T, \mathfrak{s})$ and hence $\mathfrak{n}(T, \mathfrak{s}) = \mathfrak{z} \times \mathfrak{m}$. Let $\mathfrak{u} = [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{n}(T, \mathfrak{s})$, where \mathfrak{n} denotes the Lie algebra of N .

We can assume that $\mathfrak{z} \cap \mathfrak{u} = \{0\}$. Because otherwise, according to Proposition 18, there is a one-parameter subgroup in Z which is distorted in the analytic subgroup of S associated to \mathfrak{n} , hence in S as well. Let $q : \mathfrak{z} \times \mathfrak{m} \rightarrow \mathfrak{m}$ be the projection onto the second factor. Let V' be a complement for $\mathfrak{u} \cap \mathfrak{m}$ in \mathfrak{u} , that is $(\mathfrak{u} \cap \mathfrak{m}) \oplus V' = \mathfrak{u}$. If $q(V') \cap ([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) \neq \{0\}$, we claim that there is a one-parameter subgroup in Z which is distorted in S . To see why, let $v \in V'$ such that $q(v) = x + y \neq 0$, with $x \in [\mathfrak{m}, \mathfrak{m}]$ and $y \in \mathfrak{u} \cap \mathfrak{m}$. The element $z = v - (x + y)$ belongs to \mathfrak{z} . Notice that $z \neq 0$, because otherwise we would have $v = x + y \in V' \cap (\mathfrak{u} \cap \mathfrak{m}) = \{0\}$. Similarly $x \neq 0$, because otherwise $z = v - y \in \mathfrak{u} \cap \mathfrak{z} = \{0\}$. Since $x \in [\mathfrak{m}, \mathfrak{m}]$, the one-parameter subgroup $t \mapsto \exp(tx)$ is distorted in S , according to Proposition 18. Since $v - y \in \mathfrak{u} \subseteq [\mathfrak{n}, \mathfrak{n}]$, the one-parameter subgroup $t \mapsto \exp(t(v - y)x)$ is distorted in S , according to Proposition 18. The sub-algebra of \mathfrak{s} spanned by x and z is isomorphic to \mathbb{R}^2 . Hence, as $z = (-x) + (v - y)$, the one-parameter subgroup $t \mapsto \exp(tz)$ is also distorted in S . (The geometric picture is the following. To reach the element $\exp(n^2z)$ where n is large, we start from the identity in S and we first reach with a path in $\exp([\mathfrak{m}, \mathfrak{m}])$, of length linear in n , the point $\exp(n^2(-x))$. Then we follow a path between $\exp(n^2(-x))$ and $\exp(n^2z)$, obtained as the left-translated in S by $\exp(n^2(-x))$ of a path in $\exp([\mathfrak{n}, \mathfrak{n}])$ between the identity and $\exp(n^2(v - y))$ and of length linear in n .)

Hence we assume that $q(V') \cap ([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) = \{0\}$, and we will show that \mathfrak{z} is a direct factor in \mathfrak{s} . We consider a complement W for $q(V') \oplus ([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m})$ in \mathfrak{m} . Hence

$$\mathfrak{m} = ([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) \oplus q(V') \oplus W.$$

We define the sum of subspaces,

$$\tilde{\mathfrak{m}} = ([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + V' + W.$$

We claim that $\mathfrak{z} \times \tilde{\mathfrak{m}} \cong \mathfrak{n}(T, \mathfrak{s})$. The inclusion $\mathfrak{z} \times \tilde{\mathfrak{m}} \subseteq \mathfrak{n}(T, \mathfrak{s})$ is obvious. The opposite inclusion is true because if $x \in \mathfrak{n}(T, \mathfrak{s})$, then $x = z + y$ with $z \in \mathfrak{z}$ and $y \in \mathfrak{m}$, and we can write $y = y_1 + y_2 + y_3$ with $y_1 \in [\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}$, $y_2 \in q(V')$, and $y_3 \in W$. Let $v \in V'$ such that $q(v) = y_2$. Define $z' = v - y_2 \in \mathfrak{z}$. Hence $x = z + y = (z - z') + y_1 + v + y_3$, with $z - z' \in \mathfrak{z}$, $y_1 \in [\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}$, $v \in V'$, and $y_3 \in W$. The sum is direct because,

$$\begin{aligned} \dim(\tilde{\mathfrak{m}}) &\leq \dim([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + \dim(V') + \dim(W) \\ &= \dim([\mathfrak{m}, \mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + \dim(q(V')) + \dim(W) = \dim(\mathfrak{m}). \end{aligned}$$

As the subspace $\tilde{\mathfrak{m}}$ of $\mathfrak{n}(T, \mathfrak{s})$ is a sub-algebra (because it contains $[\mathfrak{n}(T, \mathfrak{s}), \mathfrak{n}(T, \mathfrak{s})] = [\mathfrak{m}, \mathfrak{m}]$), and as \mathfrak{z} is central, we obtain a direct product as claimed. What we gained in replacing \mathfrak{m} with $\tilde{\mathfrak{m}}$, is that the latest contains \mathfrak{u} because $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{m}) \oplus V' \subseteq \tilde{\mathfrak{m}}$. This will be crucial in finishing the proof.

We have:

$$\mathfrak{s} = \mathfrak{n}(T, \mathfrak{s}) \oplus \mathfrak{i}(T, \mathfrak{s}) = \mathfrak{z} \oplus \tilde{\mathfrak{m}} \oplus \mathfrak{i}(T, \mathfrak{s}).$$

The proof will be finish if we show that $\tilde{\mathfrak{m}} \oplus \mathfrak{i}(T, \mathfrak{s})$ is a sub-algebra of \mathfrak{s} . Let $x, x' \in \tilde{\mathfrak{m}}$ and $y, y' \in \mathfrak{i}(T, \mathfrak{s})$. We have,

$$[x + y, x' + y'] = \underbrace{[x, x']}_{\in \tilde{\mathfrak{m}}} + \underbrace{[x, y'] + [y, x']}_{\in \mathfrak{i}(T, \mathfrak{s})} + [y, y'].$$

Hence we have to show that $[y, y'] \in \tilde{\mathfrak{m}} \oplus \mathfrak{i}(T, \mathfrak{s})$. As $\mathfrak{i}(T, \mathfrak{s}) \subseteq \ker(p) = \mathfrak{n}$, we have $[\mathfrak{i}(T, \mathfrak{s}), \mathfrak{i}(T, \mathfrak{s})] \subseteq [\mathfrak{n}, \mathfrak{n}]$. As \mathfrak{n} is an ideal of \mathfrak{s} , it is preserved by $ad(T)$. Hence, as $ad(T)$ is a derivation it also preserves $[\mathfrak{n}, \mathfrak{n}]$. Let $\mathfrak{n}(T, [\mathfrak{n}, \mathfrak{n}])$, resp. $\mathfrak{i}(T, [\mathfrak{n}, \mathfrak{n}])$, be the big kernel, resp. the small image, of the restriction of $ad(T)$ to $[\mathfrak{n}, \mathfrak{n}]$. We have

$$[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}(T, [\mathfrak{n}, \mathfrak{n}]) \oplus \mathfrak{i}(T, [\mathfrak{n}, \mathfrak{n}]).$$

Now we can conclude because $\mathfrak{n}(T, [\mathfrak{n}, \mathfrak{n}]) \subseteq \mathfrak{n}(T, \mathfrak{s}) \cap [\mathfrak{n}, \mathfrak{n}] = \mathfrak{u} \subseteq \tilde{\mathfrak{m}}$ and $\mathfrak{i}(T, [\mathfrak{n}, \mathfrak{n}]) \subseteq \mathfrak{i}(T, \mathfrak{s})$.

Remark 20 In fact, the proof of Proposition 19 shows that if the distortion function of the central one-parameter subgroup under consideration is not bounded, then it grows at least linearly.

Lemma 21 *Let V be a simply-connected abelian Lie group with a left-invariant Riemannian metric and let Γ be a lattice of V . Let $t \mapsto \exp(tX)$ be a one-parameter subgroup of V which projects to a dense subgroup of V/Γ . Let $\Gamma = \mathbb{Z} \oplus A$ be a splitting of Γ and let z be a generator of the \mathbb{Z} -factor. Then there are constants $C > 1$, $0 < \alpha < 1 < \beta$, such that for each $n \in \mathbb{Z}$, there exists $t \in \mathbb{R}$, satisfying $\alpha n < |t| < \beta n$ such that in the cylinder V/A equipped with the Riemannian metric locally isometric to V , we have $d(nz, \exp(tX)) \leq C$ (where nz , and $\exp(tX)$ are viewed in the cylinder V/A).*

Proof It is enough to prove the lemma in the case $V = \mathbb{R}^d$ with the usual coordinates and metric, $\Gamma = \mathbb{Z}^d$ generated by the canonical basis vectors, $t \mapsto tX$ with dense image in $\mathbb{R}^d/\mathbb{Z}^d$, $\mathbb{Z}^d = \mathbb{Z} \oplus \mathbb{Z}^{d-1}$, and z the first vector of the canonical basis. As $t \mapsto tX$ has dense image in $\mathbb{R}^d/\mathbb{Z}^d$, the vector X is not orthogonal to z . We may assume that X and z are in the same connected component of the complement of the hyper-plane orthogonal to z . Let $0 \leq \theta < \pi/2$ be the angle between X and z . We may assume $n \in \mathbb{N}$. We choose $t = z^n / \cos \theta$. In the cylinder $\frac{\mathbb{R}^d}{\{0\} \times \mathbb{Z}^{d-1}}$, we have $d(nz, tX) \leq \sqrt{2}/2$.

Proposition 22 *Let G be a connected Lie group and assume that the radical \sqrt{G} of G is not linear. Then the fundamental group of G is distorted in the universal cover of G . Also there exists a distorted central \mathbb{Z} -extension E of G with E connected.*

Proof For this proof let us set $R = \sqrt{G}$. Since R is non-linear, there exists by Lemma 13 a compact subgroup $K < R$ such that $[R, R] \cap K \neq \{e\}$. Therefore, according to Lemma 9, there is a non-trivial one-parameter subgroup $\phi : \mathbb{R} \rightarrow [R, R] \cap T_N$, where T_N is the maximal compact subgroup of the nilradical N of G , which is also the nilradical of R . The closure of $\phi(\mathbb{R})$ in T_N is a torus $T \subseteq T_N$. Notice that T_N

is central in G because T_N is a normal subgroup of G with discrete automorphism group. According to Lemma 6, the sequence of inclusions of closed connected normal subgroups $T \subseteq R \subseteq G$ induces the commutative diagram,

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \Gamma & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & \Gamma \cap \tilde{R} & \longrightarrow & \tilde{R} & \longrightarrow & R & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & \Gamma \cap \tilde{T} & \longrightarrow & \tilde{T} & \longrightarrow & T & \longrightarrow & 1,
 \end{array}$$

where \tilde{G} , \tilde{R} , and \tilde{T} , are resp. the universal covers of G , R , and T , where Γ , $\Gamma \cap \tilde{R}$, and $\Gamma \cap \tilde{T}$, are central discrete subgroups, resp. isomorphic to the fundamental groups of G , R , and T , and where the vertical arrows are injections of closed normal subgroups. Let $\tilde{\phi}(t) = \exp(tX)$ be the one-parameter subgroup of \tilde{T} which covers ϕ . Let \mathfrak{t} denote the Lie algebra of \tilde{R} . The vector $X \neq 0$ belongs to $[\mathfrak{t}, \mathfrak{t}]$ and is central in the Lie algebra of \tilde{G} , hence we can apply Proposition 19, with $S = \tilde{R}$, $N = [S, S]$ and Z the one-parameter subgroup $\tilde{\phi}$ in \tilde{T} , to deduce that the one-parameter subgroup $\tilde{\phi}$ is distorted in \tilde{R} . As the image of $\tilde{\phi}$ in $\tilde{T} \cong \mathbb{R}^{\dim(T)}$ is a line and as $\Gamma \cap \tilde{T} \cong \mathbb{Z}^{\dim(T)}$ is cocompact in \tilde{T} , we deduce that $\Gamma \cap \tilde{T}$ is distorted in \tilde{R} , hence in \tilde{G} as well. As $\Gamma \cap \tilde{T}$ is undistorted in Γ , this shows that Γ is distorted in \tilde{G} .

To prove the existence of a distorted topological \mathbb{Z} -extension of G , let us choose a non-zero element z_0 in the free abelian group $\Gamma \cap \tilde{T}$. It is possible to choose a direct summand \mathbb{Z} in the finitely generated abelian group Γ , such that a generator z of the direct summand \mathbb{Z} shares a non-zero power with z_0 : there exist $m, n \in \mathbb{N}$, such that $z_0^m = z^n$. This implies that the infinite cyclic groups $\langle z_0 \rangle$ and $\langle z \rangle$ lie at bounded distance from each other in \tilde{G} (with respect to any left-invariant Riemannian metric on \tilde{G}). Let B be a complement of $\langle z \rangle = \mathbb{Z}$ in Γ and let $E = \tilde{G}/B$. Let $p : \tilde{G} \rightarrow E$ denote the canonical projection. The connected Lie group E is a topological \mathbb{Z} -extension of G , with kernel generated by the image $p(z) \in E$. The proof will be finished if we prove that the kernel $\langle p(z) \rangle \cong \mathbb{Z}$ is distorted in E . As \tilde{T} is the universal cover of the closure T of ϕ , and as $\tilde{\phi}$ is distorted in \tilde{G} , we deduce from Lemma 17 that the embedding of $p(\tilde{\phi})$ in E is distorted in E , and from Lemma 21 (applied with $\langle p(z_0) \rangle = \mathbb{Z}$) that $\langle p(z) \rangle = \mathbb{Z}$ itself is distorted in E .

Remark 23 The above proof shows, under the hypothesis of Proposition 22, that the distortion function of the fundamental group in the universal cover of G grows at least linearly. Similarly, the kernel of the distorted topological \mathbb{Z} -extension E of G is at least linearly distorted.

Remark 24 In the example of Remark 12, the fundamental group $\pi_1(G)$ is distorted in the universal cover \tilde{G} of G , but each infinite cyclic subgroup of $\pi_1(G)$ is undistorted in \tilde{G} . This last condition implies that the image of the one-parameter subgroup ϕ (in the proof of Proposition 22) is not closed.

3 On Borel cohomology

3.1 The relationship between the various cohomology groups

In this subsection we recall the relationship we need between various cohomology groups. Recall that a map $f : X \rightarrow Y$ of topological spaces, is *Borel* if it is measurable with respect to the σ -algebras generated by the open subsets of X resp. Y . For G any topological group and A a metric abelian group, we write $C_B^n(G, A)$, resp. $C_{Bb}^n(G, A)$ for the group of Borel maps $G^n \rightarrow A$, resp. bounded Borel maps. With the usual differentials, this defines a cochain complexes $C_{B(b)}^*(G, A)$, whose respective cohomologies,

$$H_B^*(G, A) \text{ and } H_{Bb}^*(G, A)$$

are the *Borel*, resp. *Borel bounded* cohomology of G with coefficients in A ; if A is finitely generated abelian, we always assume that the metric on A corresponds to the word metric coming from a finite symmetric generating set. In a similar way, the groups H_c^* , H_{cb}^* , H_b^* are defined using cochains which are continuous, continuous bounded, or just bounded. We refer the reader to [29–31], [20], and [28] for the definition and functorial properties of these cohomology theories.

All groups will usually be supposed to be separable and locally compact, with topology given by a complete metric (occasionally we will also consider non-separable groups like \mathbb{R}^δ).

Our main object of study is the forgetful map $H_{Bb}^*(G, \mathbb{Z}) \rightarrow H_B^*(G, \mathbb{Z})$ for the case of a virtually connected Lie group. Note that the target group $H_B^*(G, \mathbb{Z})$ is naturally isomorphic to the singular cohomology $H^*(BG, \mathbb{Z})$ of the classifying space BG of G (cf. [36, Theorem 4, p. 93] for the proof and [1, Sect. 7] for some useful comments). There are also canonical isomorphisms $H_c^*(G, \mathbb{R}) \cong H_B^*(G, \mathbb{R})$ (see [36, Theorem 3, p. 91] and [30, Sect. 7. (2), p. 32]), and $H_{cb}^*(G, \mathbb{R}) \cong H_{Bb}^*(G, \mathbb{R})$ (see [10, Sect. 2.3, (2i), p. 15] which refers to [2, Sect. 4]). We write G^δ for G considered as a discrete group. We have a canonical isomorphism $H^*(G^\delta, \mathbb{R}) \cong H^*(BG^\delta, \mathbb{R})$. The relationship between the various cohomology groups can be expressed by the following natural commutative diagram:

$$\begin{array}{ccccc}
 H_{Bb}^*(G, \mathbb{Z}) & \longrightarrow & H_B^*(G, \mathbb{Z}) \cong H^*(BG, \mathbb{Z}) & \longrightarrow & H^*(BG, \mathbb{R}) \\
 \downarrow & & & & \downarrow \phi \\
 & & H_{cb}^*(G, \mathbb{R}) \cong H_{Bb}^*(G, \mathbb{R}) & \longrightarrow & H_c^*(G, \mathbb{R}) \cong H_B^*(G, \mathbb{R}) \\
 & & \downarrow & & \downarrow \\
 H_b^*(G^\delta, \mathbb{Z}) & \longrightarrow & H_b^*(G^\delta, \mathbb{R}) & \longrightarrow & H^*(G^\delta, \mathbb{R}).
 \end{array}$$

The map ϕ from the upper right corner is defined as the composition:

$$\phi : H^*(BG, \mathbb{R}) \cong H^*(BG, \mathbb{Z}) \otimes \mathbb{R} \cong H_B^*(G, \mathbb{Z}) \otimes \mathbb{R} \rightarrow H_B^*(G, \mathbb{R}). \tag{1}$$

3.2 On Borel cohomology

In this subsection, we recall and prove several general properties in bounded cohomology (Lemmata 25, 26, 29, 33, Corollaries 28, 30, Proposition 31). We also give examples of unbounded characteristic classes of any even degree (Example 37).

Lemma 25 *Let G be a topological group and $x \in H_B^d(G, \mathbb{Z})$ be such that for some $n > 0$, nx is bounded. Then x is bounded as well.*

Proof The short exact coefficients sequence, $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$, yields long exact cohomology sequences, for Borel bounded cohomology as well as for Borel cohomology:

$$\begin{array}{ccccccc}
 H_{Bb}^{d-1}(G, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H_{Bb}^d(G, \mathbb{Z}) & \xrightarrow{n} & H_{Bb}^d(G, \mathbb{Z}) & \longrightarrow & H_{Bb}^d(G, \mathbb{Z}/n\mathbb{Z}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_B^{d-1}(G, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H_B^d(G, \mathbb{Z}) & \xrightarrow{n} & H_B^d(G, \mathbb{Z}) & \longrightarrow & H_B^d(G, \mathbb{Z}/n\mathbb{Z}).
 \end{array}$$

Using that the vertical maps are isomorphisms for $\mathbb{Z}/n\mathbb{Z}$ -coefficients, the result follows from a simple diagram chase.

Lemma 26 *Let G be a topological group. Let x^δ be a characteristic class of G in the image of the composition $H^d(BG, \mathbb{Z}) \rightarrow H^d(BG, \mathbb{R}) \rightarrow H^d(BG^\delta, \mathbb{R})$. If x^δ is bounded then it admits a representative cocycle whose set of values on the singular simplices of BG^δ is a finite subset of \mathbb{Z} .*

Proof The exact sequence of coefficients $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1 \rightarrow 0$ admits a bounded section. Let us focus on the following two horizontal exact sequences with commutative squares, it induces.

$$\begin{array}{ccccc}
 H_b^d(BG^\delta, \mathbb{Z}) & \longrightarrow & H_b^d(BG^\delta, \mathbb{R}) & \longrightarrow & H_b^d(BG^\delta, S^1) \\
 \downarrow & & \downarrow & & \downarrow = \\
 H^d(BG^\delta, \mathbb{Z}) & \longrightarrow & H^d(BG^\delta, \mathbb{R}) & \longrightarrow & H^d(BG^\delta, S^1).
 \end{array}$$

By hypothesis, x^δ has a representative cocycle with integer values, hence it goes to zero in $H^d(BG^\delta, S^1)$. By hypothesis, it is the image of an element $w \in H_b^d(BG^\delta, \mathbb{R})$. A simple diagram chase shows that w is in the image of $H_b^d(BG^\delta, \mathbb{Z}) \rightarrow H_b^d(BG^\delta, \mathbb{R})$.

Lemma 27 *Let $p : H \rightarrow G$ be a covering map of connected Lie groups. Then the induced map $H^*(BG, \mathbb{R}) \rightarrow H^*(BH, \mathbb{R})$ is surjective.*

Proof As p is a $\pi_1(G)$ -Galois covering and as $\pi_1(G)$ is finitely generated abelian, we can factor p in a sequence of connected covering spaces $H = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n = G$ with $X_i \rightarrow X_{i+1}$ a cyclic covering. If $X_i \rightarrow X_{i+1}$ is a finite covering,

the induced map $H^*(BX_{i+1}, \mathbb{R}) \rightarrow H^*(BX_i, \mathbb{R})$ is an isomorphism, because the fiber of $BX_i \rightarrow BX_{i+1}$ is \mathbb{R} -acyclic. In case $X_i \rightarrow X_{i+1}$ is an infinite cyclic covering, there is a circle fibration $S^1 \rightarrow BX_i \rightarrow BX_{i+1}$ with associated Gysin sequence with \mathbb{R} -coefficients

$$H^d(BX_{i+1}) \xrightarrow{\cup e} H^{d+2}(BX_{i+1}) \longrightarrow H^{d+2}(BX_i) \xrightarrow{\theta(d)} H^{d+1}(BX_{i+1})$$

in which $\theta(d)$ is the zero map, because the real cohomology of a connected Lie group is concentrated in even dimensions. It follows that $H^*(BX_{i+1}, \mathbb{R}) \rightarrow H^*(BX_i, \mathbb{R})$ is surjective and therefore the composite map $H^*(BG, \mathbb{R}) \rightarrow H^*(BH, \mathbb{R})$ is surjective too.

This yields the following useful corollary.

Corollary 28 *Let $p : H \rightarrow G$ be a covering map of connected Lie groups. If all Borel cohomology classes in degree d with \mathbb{Z} coefficients for G are bounded then they are all bounded for H as well.*

Proof Using the natural isomorphism

$$H_B^*(L, \mathbb{Z}) \cong H^*(BL, \mathbb{Z})$$

for L a connected Lie group, we have a natural commutative diagram

$$\begin{CD} H_{Bb}^d(G, \mathbb{Z}) @>p_{Bb}^*>> H_{Bb}^d(H, \mathbb{Z}) \\ @VVV @VVV \\ H_B^d(G, \mathbb{Z}) = H^d(BG, \mathbb{Z}) @>p_B^*>> H_B^d(H, \mathbb{Z}) = H^d(BH, \mathbb{Z}). \end{CD}$$

In view of Lemma 27 we know that p_B^* maps onto a subgroup of finite index. Therefore, if all Borel cohomology classes in degree d for G are bounded, then for every $x \in H_B^d(H, \mathbb{Z})$ there is an $m > 0$ so that mx is bounded. But this implies by Lemma 25 that already x is bounded.

Lemma 29 *Let G be a topological group. Suppose that the image of $x \in H_B^*(G, \mathbb{Z})$ in $H_B^*(G, \mathbb{R})$ is bounded, (i.e., lies in the image of $H_{Bb}^*(G, \mathbb{R}) \rightarrow H_B^*(G, \mathbb{R})$). Then x is bounded as well.*

Proof The short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1 \rightarrow 0$ of topological groups admits a Borel bounded section $S^1 \rightarrow \mathbb{R}$ and gives therefore rise to long exact cohomology sequences and a commutative diagram

$$\begin{CD} \cdots @>>> H_{Bb}^{d-1}(G, S^1) @>>> H_{Bb}^d(G, \mathbb{Z}) @>>> H_{Bb}^d(G, \mathbb{R}) @>>> H_{Bb}^d(G, S^1) @>>> \cdots \\ @. @VVV @VVV @VVV @VVV \\ \cdots @>>> H_B^{d-1}(G, S^1) @>>> H_B^d(G, \mathbb{Z}) @>>> H_B^d(G, \mathbb{R}) @>>> H_B^d(G, S^1) @>>> \cdots \end{CD}$$

A simple diagram chase completes the proof.

Corollary 30 *If G is a compact Lie group, then all elements of $H_B^*(G, \mathbb{Z})$ are bounded.*

Proof Let $x \in H_B^*(G, \mathbb{Z})$. Then its image $y \in H_B^*(G, \mathbb{R})$ is bounded, because by van Est's theorem, the continuous cohomology of a compact Lie group vanishes in positive dimensions, and $H_B^*(G, \mathbb{R}) = H_c^*(G, \mathbb{R})$ by Wigner [36, Theorem 3, p. 91].

Proposition 31 *Let G be a virtually connected Lie group and let G^0 be the connected component of the identity. If $x \in H_B^*(G, \mathbb{Z})$ restricts to a bounded class in $H_B^*(G^0, \mathbb{Z})$ then x itself is bounded. If $y \in H^*(BG^\delta, \mathbb{R})$ restricts to a class in $H^*(B(G^0)^\delta, \mathbb{R})$ which has a representing cocycle which takes only finitely many values then y has such a representative too.*

Proof Assume $x \in H_B^*(G, \mathbb{Z})$ restricts to a bounded class in $H_B^*(G^0, \mathbb{Z})$. It suffices by Lemma 29 to show that the image $u = \gamma(x) \in H_B^*(G, \mathbb{R})$ of x is bounded. We have a natural commutative diagram with horizontal arrows β and δ induced by restriction:

$$\begin{array}{ccccc}
 H_{Bb}^d(G, \mathbb{Z}) & \xrightarrow{\alpha} & H_{Bb}^d(G, \mathbb{R}) & \xrightarrow{\beta} & H_{Bb}^d(G^0, \mathbb{R})^{G/G^0} \\
 \downarrow & & \downarrow \phi & & \downarrow \psi|_{\text{inv}} \\
 x \in H_B^d(G, \mathbb{Z}) & \xrightarrow{\gamma} & H_B^d(G, \mathbb{R}) & \xrightarrow{\delta} & H_B^d(G^0, \mathbb{R})^{G/G^0}.
 \end{array}$$

That the horizontal arrows β and δ are isomorphisms follows from the Lyndon-Hochschild-Serre spectral sequences for the short exact sequences $G^0 \rightarrow G \rightarrow G/G^0$, using that the Borel bounded cohomology and the Borel cohomology with \mathbb{R} coefficients vanishes for the finite group G/G^0 in positive dimensions (for the Lyndon-Hochschild-Serre spectral sequence in continuous, resp. continuous bounded, cohomology see [2] resp. [28, Chap. IV, Sect. 12]; as we have mentioned at the beginning of Sect. 3, there are natural isomorphisms $H_B^*(G, \mathbb{R}) \cong H_c^*(G, \mathbb{R})$, resp. $H_{Bb}^*(G, \mathbb{R}) \cong H_{cb}^*(G, \mathbb{R})$). From our assumption it follows that $\delta(u) = v$ is bounded, say $v = \psi(w)$ for some $w \in H_{Bb}^d(G^0, \mathbb{R})$. By averaging with respect to the G/G^0 -action we can form $\bar{w} = \frac{1}{|G:G^0|} \sum gw$ where the sum is taken over a set of coset representatives of G^0 in G . Then $\bar{w} \in H_{Bb}^d(G^0, \mathbb{R})^{G/G^0}$ and $\psi(\bar{w}) = v$. Thus $\beta^{-1}(\bar{w})$ is a bounded representative for u and we are done with the first case.

The case of $H^*(BG^\delta, \mathbb{R})$ can be dealt with in a similar way, using the fact that the restriction map induces an isomorphism $H^*(BG^\delta, \mathbb{R}) \rightarrow H^*(B(G^0)^\delta, \mathbb{R})^{G/G^0}$.

Example 32 The following is an example of a virtually connected Lie group G with all cohomology classes in $H_B^2(G, \mathbb{Z})$ bounded but having an unbounded class in $H_B^2(G^0, \mathbb{Z})$. Take \mathbf{H} to be the three dimensional real Heisenberg group. It admits an involution $T : \mathbf{H} \rightarrow \mathbf{H}$ which induces multiplication by -1 on the center of \mathbf{H} . In matrix notation,

$$T : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -a & -c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Then T preserves the infinite cyclic central subgroup \mathbb{Z} generated by any chosen non-zero central element. Thus T induces an involution on \mathbf{H}/\mathbb{Z} , which induces multiplication by -1 on $H^2(B(\mathbf{H}/\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$. It follows that the semi-direct product $G := (\mathbf{H}/\mathbb{Z}) \rtimes_T \mathbb{Z}/2\mathbb{Z}$ has $H_B^2(G, \mathbb{Z})$ finite, hence bounded according to Lemma 25 (it is a finite group isomorphic to $H^2(BO(2), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, because $H_B^2(G, \mathbb{Z}) \cong H^2(BG, \mathbb{Z})$ and G has $O(2)$ as a maximal compact subgroup). Now let $x \in H_B^2(G^0, \mathbb{Z}) \cong H^2(BG^0, \mathbb{Z}) \cong \mathbb{Z}$ be a generator. We claim that x is not bounded. If x were bounded, all of $H_B^*(G^0, \mathbb{Z})$ were bounded, because $H_B^*(G^0, \mathbb{Z})$ is generated by an algebra by x . But this contradicts Theorem 1, because the closed subgroup $[\sqrt{G^0}, \sqrt{G^0}] \cong \mathbb{R}/\mathbb{Z}$ is not simply-connected and thus $\sqrt{G^0} = G^0$ is not linear.

Lemma 33 *Let A be a finitely generated abelian group. If $f : G \rightarrow Q$ is a homomorphism of Lie groups and a homotopy equivalence of the underlying topological spaces, then the induced map $f^* : H_B^*(Q, A) \rightarrow H_B^*(G, A)$ is an isomorphism. In particular, if all classes of $H_B^*(Q, A)$ are bounded, then the same is true for $H_B^*(G, A)$.*

Proof The map f induces a homotopy equivalence of classifying spaces $Bf : BG \rightarrow BQ$. Indeed, it induces an isomorphism at the level of homotopy groups and the spaces in question have the homotopy type of CW -complexes. On the other hand, $H_B^*(G, A)$ is naturally isomorphic to $H^*(BG, A)$ (cf. [36]). This proves that f^* is an isomorphism. If c is a bounded representative of $[c] \in H_B^*(Q, A)$, then f^*c is a bounded representative of $f^*[c]$. This concludes the proof.

For the convenience of the reader we give the proof of the following two results from Bucher-Karlsson's thesis.

Proposition 34 (Bucher-Karlsson [8, p. 60]) *Let G be a locally compact group with a cocompact lattice Γ . Let $res : H_c^*(G, \mathbb{R}) \rightarrow H^*(\Gamma, \mathbb{R})$ be the restriction map and let $x \in H_c^*(G, \mathbb{R})$. If $res(x)$ is bounded then x admits a continuous, bounded cocycle representative.*

Proof There is a commutative diagram:

$$\begin{array}{ccc}
 H_{cb}^*(G, \mathbb{R}) & \xrightarrow{F} & H_c^*(G, \mathbb{R}) \\
 \uparrow tr & & \uparrow tr \\
 H_b^*(\Gamma, \mathbb{R}) & \xrightarrow{F} & H^*(\Gamma, \mathbb{R}),
 \end{array}$$

where F denotes forgetful maps from bounded cohomologies and tr denotes the transfer maps (cf. [28, p. 107]). The equality

$$tr \circ res = id,$$

holds on $H_c^*(G, \mathbb{R})$ (as well as on $H_{cb}^*(G, \mathbb{R})$). By hypothesis, there exists $y \in H_b^*(\Gamma, \mathbb{R})$ such that $F(y) = res(x)$. A continuous, bounded cocycle representative of the transfer of y exists by definition and

$$F(tr(y)) = tr(F(y)) = tr(res(x)) = x.$$

The map $\phi : H^*(BG, \mathbb{R}) \rightarrow H_c^*(G, \mathbb{R})$ in the following Proposition is the one defined in Subsect. 3.1, Eq. 1.

Proposition 35 (Bucher-Karlsson [8, p. 60].) *Let G be a Lie group of the form $G_a(\mathbb{R})^0$ for some semi-simple linear algebraic \mathbb{R} -group G_a . Every element in the image of $\phi : H^*(BG, \mathbb{R}) \rightarrow H_c^*(G, \mathbb{R})$ admits a continuous, bounded cocycle representative.*

Proof Let $x \in H^*(BG, \mathbb{R})$. Its image $\phi(x)$ maps to the element $x^\delta \in H^*(BG^\delta, \mathbb{R}) \cong H^*(G^\delta, \mathbb{R})$. The proof of Gromov's result [17, Sect. 1.3, p. 23] by Bucher-Karlsson [9] works in a semi-algebraic setting, hence applies to $G = G_a(\mathbb{R})^0$. Therefore, the class x^δ is bounded. According to [3], there exists a cocompact lattice Γ in G . As the arrows of the following commutative diagram,

$$\begin{array}{ccc}
 H_c^*(G, \mathbb{R}) & & \\
 \downarrow & \searrow^{res} & \\
 H_b^*(G^\delta, \mathbb{R}) & \xrightarrow{res} & H^*(\Gamma, \mathbb{R}),
 \end{array}$$

preserve boundedness, the class $res(\phi(x)) \in H^*(\Gamma, \mathbb{R})$ is also bounded. Proposition 34 shows that the class $\phi(x)$ has a continuous, bounded representative.

Translating this into the context of Borel cohomology and passing to Lie groups with finitely many connected components, we obtain the following.

Corollary 36 *Let G be a virtually connected Lie group with G^0 semi-simple (not necessarily linear). Then $H_{Bb}^*(G, \mathbb{Z}) \rightarrow H_B^*(G, \mathbb{Z})$ is surjective.*

Proof We first reduce to the case of the connected component G^0 by Proposition 31. Dividing by the center, we reduce to the case of $G^0/Z(G^0)$ by using Corollary 28. Now $G^0/Z(G^0) := H$ is a connected linear semi-simple Lie group. According to [25] or [37, Proposition 3.1.6], H is as a Lie group isomorphic to $G_a(\mathbb{R})^0$ for some connected real linear algebraic group G_a . Let $x \in H_B^*(H, \mathbb{Z})$. According to Lemma 29, it is enough to show that the image $y \in H_B^*(H, \mathbb{R})$ of x is bounded. In view of the commutative diagram of Subsect. 3.1, and using 35, the class y admits a continuous, bounded cocycle representative.

The following family of examples of Lie groups with non-linear radical shows that one can have unbounded primary characteristic classes in any positive (even) degree. (The basic idea in the construction appears in [15] and also in [19, Remarques 7.5, b].)

Example 37 Let G_i be the i -fold cartesian product of Heisenberg quotients \mathbf{H}/\mathbb{Z} , where \mathbf{H} is the 3-dimensional Heisenberg group and $\mathbb{Z} < \mathbf{H}$ a central subgroup. Note that the maximal compact subgroup of \mathbf{H}/\mathbb{Z} is S^1 so that $H^*(BG_i, \mathbb{R})$ is a polynomial algebra generated by 2-dimensional classes $x_j, 1 \leq j \leq i$, with the property that the map induced by the j -th injection $\mathbf{H}/\mathbb{Z} \rightarrow G_i$ maps x_j to a non-zero element in $H^2(B(\mathbf{H}/\mathbb{Z}), \mathbb{R})$. According to Goldman [15] there is a flat \mathbf{H}/\mathbb{Z} -bundle over the 2-torus T^2 given by a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbf{H}/\mathbb{Z}$ such that

$\phi^* : H^2(B(\mathbf{H}/\mathbb{Z}), \mathbb{R}) \rightarrow H^2(T^2, \mathbb{R})$ is non-zero. Taking i -fold products, we get a flat G_i -bundle of the $2i$ -torus T^{2i} , with associated map $H^*(BG_i, \mathbb{R}) \rightarrow H^*(T^{2i}, \mathbb{R})$ mapping the product $y_d := x_1 \cdots x_d$ non-trivially into $H^{2d}(T^{2i}, \mathbb{R})$, where $1 \leq d \leq i$. It follows that the images $y_d^\delta \in H^{2d}(BG_i^\delta, \mathbb{R})$ of y_d are all non-trivial. But they cannot have bounded cocycle representatives, because the bounded cohomology of the nilpotent group G_i^δ vanishes in positive dimensions. It also follows that for all $i \geq 1$ and all $1 \leq d \leq i$,

$$H_{Bb}^{2d}(G_i, \mathbb{Z}) \rightarrow H_B^{2d}(G_i, \mathbb{Z})$$

is not surjective.

3.3 Topological A -extensions

In this subsection, we define *topological A -extensions*. Moore's theorem (see Theorem 38) shows that they are closely related to 2-dimensional Borel cohomology. In Lemmata 39, 44, 45, we establish some properties of 2-cocycles associated to central extensions. In Proposition 47 we show that the 2-dimensional Borel cohomology with \mathbb{Z} -coefficients of a Lie group G is bounded if and only if the class defined by the universal cover of G is bounded. Proposition 49 shows that for a connected normal closed subgroup N of a connected Lie group G , each real class of degree 2 on the classifying space BN is the restriction of a class on BG .

A central extension of topological groups $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ will be referred to as a *topological A -extension*. In this setting, we always assume that the groups are locally compact and second countable and that the monomorphism $A \rightarrow E$ has closed image. We write $\text{Ext}_{\text{top}}(G, A)$ for the group of isomorphism classes of topological A -extensions of G , where two extensions $0 \rightarrow A \rightarrow E_1 \rightarrow G \rightarrow 1$ and $0 \rightarrow A \rightarrow E_2 \rightarrow G \rightarrow 1$ are called *isomorphic* if there is an isomorphism of topological groups $\varphi : E_1 \rightarrow E_2$ making the following diagram commute:

$$\begin{array}{ccccccc}
 & & & E_1 & & & \\
 & & & \nearrow & \downarrow \varphi & \searrow & \\
 0 & \longrightarrow & A & \longrightarrow & E_2 & \longrightarrow & G \longrightarrow 1.
 \end{array}$$

In case G is a connected Lie group and A is discrete, a topological A -extension is just a covering space $E \rightarrow G$ with covering transformation group A . Such an E is determined by a homomorphism $\phi : \pi_1(G) \rightarrow A$, with

$$E = A \times_\phi \tilde{G} = (A \times \tilde{G})/\pi_1(G)$$

with \tilde{G} the universal cover of G , where $\pi_1(G)$ acts on \tilde{G} via deck transformations and on A via ϕ . It follows that there is a natural isomorphism

$$\text{Ext}_{\text{top}}(G, A) \cong \text{Hom}(\pi_1(G), A).$$

A topological A -extension $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ always admits a Borel section σ (cf. [24]). We denote $c_\sigma : G \times G \rightarrow A$ the Borel 2-cocycle

$$c_\sigma(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}.$$

The following statement is a special case of [30, Theorem 10]:

Theorem 38 (Moore [30, Theorem 10].) *Let G and A be locally compact separable groups with A abelian. The map*

$$\begin{aligned} \text{Ext}_{\text{top}}(G, A) &\rightarrow H_B^2(G, A) \\ \{0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1\} &\mapsto [c_\sigma], \end{aligned}$$

is an isomorphism.

Lemma 39 *Let G and A be locally compact separable groups with A abelian. Let $c : G \times G \rightarrow A$ be a Borel 2-cocycle. There is a topological A -extension*

$$0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$$

and a Borel section $\sigma : G \rightarrow E$ such that $c = c_\sigma$.

Proof Let E a topological A -extension which represents $[c]$ via Moore's Theorem 38 and choose a Borel section τ of E then $[c] = [c_\tau]$. There exists a Borel map $b : G \rightarrow A$, such that $c = c_\tau + db$. Recall that the coboundary operator on a 1-cochain b , with A the trivial G -module, is $db(g, g') = b(g') - b(gg') + b(g)$. The Borel cocycle c_σ , associated to the Borel section $\sigma : G \rightarrow E$, defined as $\sigma(g) = \tau(g)b(g)$, is equal to c .

Lemma 40 *Let $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$, be an exact sequence of second countable locally compact groups, with A discrete in E . Let σ be a Borel section of the projection $p : E \rightarrow G$. Let μ_E be a Haar measure on E . There exists a Borel subset X of E with the following properties:*

1. X is relatively compact,
2. $\mu_E(X) > 0$,
3. $X = X^{-1}$,
4. $\sigma(p(X))$ is relatively compact.

Proof Let U be an open symmetric relatively compact subset of E , small enough such that the covering projection p restricted to U , is a homeomorphism onto its image $V = p(U)$. Let μ_E , resp. μ_G , be the Haar measure on E , resp. on G , such that $\mu_E(U) = 1$, resp. such that $\mu_G(V) = 1$. The uniqueness of the Haar measure, together with the fact that the homomorphism p is a local homeomorphism, imply that for any Borel subset A of U , we have $\mu_E(A) = \mu_G(p(A))$ (see [5, Intégration, chapitre 7, paragraphe 2, numéro 7, proposition 10, p. 60]). Let $E = \cup_{n \in \mathbb{N}} K_n$, be an exhaustion of E by compact subsets, such that $K_n \subseteq K_{n+1}$. Let $B_n = \{v \in V : \sigma(v) \in K_n\} = V \cap \sigma^{-1}(K_n)$.

It is a Borel subset of G because σ is a Borel map. Notice that

$$\lim_{n \rightarrow \infty} \mu_G(V \setminus B_n) = 0.$$

We conclude from this, and from the fact that the modular function of G is bounded on V , that there exists $N \in \mathbb{N}$, such that $\mu_G(V \setminus B_N) < 1/2$, and such that $\mu_G((V \setminus B_N)^{-1}) < 1/2$. Let

$$Y = V \setminus ((V \setminus B_N) \cup (V \setminus B_N)^{-1}).$$

The set Y has the following properties:

1. Y is relatively compact,
2. $\mu_G(Y) > 0$,
3. $Y = Y^{-1}$,
4. $\sigma(Y) \subseteq K_N$.

Let $X = p^{-1}(Y) \cap V$. The set X has the required properties.

Remark 41 A projection of separable locally compact groups always admits a locally bounded Borel section [24, Lemma 2]. The point in the above lemma is that the given Borel section σ is not assumed to be locally bounded.

The following proposition has first been proved by Gersten [14], in the setting of finitely generated groups.

Proposition 42 *Let $0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$, be an exact sequence of Lie groups, with E connected, A finitely generated discrete (hence central) in E . Let σ be a Borel section of the projection $p : E \rightarrow G$, with $\sigma(e) = e$. Let $c : G \times G \rightarrow A$, $c(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}$ be the corresponding cocycle. If c is bounded then A is undistorted in E .*

Proof Let $x \in E$. There exist a unique element $a = a(x) \in A$ and a unique element $g = g(x) \in G$ such that $x = a\sigma(g)$. Let $X \subseteq E$ be as in Lemma 40. The set $X(\sigma(p(X)))^{-1}$, is relatively compact in E . This implies, together with the hypothesis that A is discrete in E , that there exists a finite symmetric generating set S of A with the property that if $x \in X$, then $a(x) \in S$.

As $\mu_E(X) > 0$, the set $XX^{-1} = XX$ contains a neighborhood of the identity of E (see [21, Chap. XII, Sect. 61, Exercise 3, p. 268]). As E is connected, the set X generates E .

Let $a \in A$ of length relative to X equal to n . Hence, there exist $x_1, \dots, x_n \in X$ such that

$$\begin{aligned} a &= x_1 \dots x_n = a_1\sigma(g_1) \dots a_n\sigma(g_n) \\ &= a_1 \dots a_n c(g_1, g_2) \dots c(g_1 \dots g_{n-1}, g_n)\sigma(g_1 \dots g_n). \end{aligned}$$

Applying p to the above equality, we get

$$e = p(a_1 \dots a_n c(g_1, g_2) \dots c(g_1 \dots g_{n-1}, g_n)\sigma(g_1 \dots g_n)) = g_1 \dots g_n.$$

Since $\sigma(e) = e$, we see that the length of a relative to S is bounded by $n + (n - 1) \sup\{L_S(c(g, g')) | L_{p(X)}(g) < n, L_{p(X)}(g') \leq 1\}$.

Remark 43 The proof of Proposition 42 shows in fact that the distortion of A in E is bounded by the growth of c .

Lemma 44 *Let $p : G \rightarrow Q$ and $\pi : \tilde{Q} \rightarrow Q$ be surjective homomorphisms between topological groups.*

1. *The total space of the pull-back $p^*(\tilde{Q}) \rightarrow G$ of $\tilde{Q} \rightarrow Q$ is a closed subgroup of $G \times \tilde{Q}$ and there is a short exact sequence of topological groups*

$$1 \rightarrow \ker(p) \rightarrow p^*(\tilde{Q}) \rightarrow \tilde{Q} \rightarrow 1.$$

2. *Assume that both $\ker \pi$ in \tilde{Q} and $\ker(p)$ in G are central. If the extension*

$$0 \rightarrow \ker(\pi) \rightarrow \tilde{Q} \rightarrow Q \rightarrow 1$$

can be defined by a Borel bounded cocycle, then the same is true for the extension

$$0 \rightarrow \ker(\pi) \rightarrow p^*(\tilde{Q}) \rightarrow G \rightarrow 1.$$

Proof To prove (1), recall that by definition,

$$p^*(\tilde{Q}) = \{(g, \tilde{q}) \in G \times \tilde{Q} \text{ such that } p(g) = \pi(\tilde{q})\}.$$

We embed $\ker(p)$ in $p^*(\tilde{Q})$ by composing the inclusions $\ker(p) \subset G \subset G \times \{e\}$. The map $\tilde{p} : p^*(\tilde{Q}) \rightarrow \tilde{Q}$, defined as $(g, \tilde{q}) \mapsto \tilde{q}$, gives the wanted exact sequence. To prove (2), we apply Lemma 39 in order to obtain a Borel section $\sigma : Q \rightarrow \tilde{Q}$ of π such that the associated cocycle c_σ is bounded. Then $\sigma^* : G \rightarrow p^*(\tilde{Q})$ defined as $g \mapsto (g, \sigma p(g))$, is a Borel section of the projection $p^*(\tilde{Q}) \rightarrow G$ and its associated cocycle satisfies

$$c_{\sigma^*}(g, g') = (e, c_\sigma(p(g), p(g'))).$$

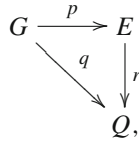
Lemma 45 *Let $p : X \rightarrow Y$ and $q : Y \rightarrow Z$ be two surjective homomorphisms of groups with central kernels. Let σ be a section of p , and τ one for q . Assume that the associated cocycles c_σ and c_τ have finite range. If the image under σ of the kernel of q is central in X then the cocycle $c_{\sigma\tau}$ associated to the section $\sigma\tau$ of qp has finite range.*

Proof Since $\sigma(\ker(q))$ is central, the kernel of qp is central so that $qp : X \rightarrow Z$ is a central $\ker(qp)$ extension. One verifies that

$$c_{\sigma\tau}(x, y) = c_\sigma(\tau(x), \tau(y)) \cdot \sigma(c_\tau(x, y)) \cdot [c_\sigma(c_\tau(x, y), \tau(xy))]^{-1}.$$

Thus $c_{\sigma\tau}$ is a product of three functions each of which takes only finitely many values. It follows that $c_{\sigma\tau}$ has finite range.

Remark 46 Consider the following commutative diagram of groups



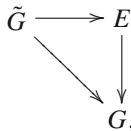
with q and r surjective. If σ is a section of q then $p\sigma$ is a section of r and

$$\begin{aligned}
 c_{p\sigma}(x, y) &= p\sigma(x)p\sigma(y) (p\sigma(xy))^{-1} \\
 &= p \left(\sigma(x)\sigma(y) (\sigma(xy))^{-1} \right) = (p \circ c_\sigma)(x, y).
 \end{aligned}$$

Proposition 47 *Let G be connected Lie group. The following conditions are equivalent.*

1. *The class in $H^2_B(G, \pi_1(G))$, corresponding (in the sense of Theorem 38) to the universal cover of G , is bounded.*
2. *All classes in $H^2_B(G, \mathbb{Z})$ are bounded.*
3. *For every finitely generated abelian group A , all classes in $H^2_B(G, A)$ are bounded.*

Proof We first prove that (1) implies (3). Let $x \in H^2_B(G, A)$ and let E be a central extension of G with central discrete kernel A corresponding to x . Let E^0 be the connected component of E . The restriction of the projection $E \rightarrow G$ to E^0 is a connected cover of G . Hence, we obtain a commutative diagram of covering groups,



where \tilde{G} is the universal cover of G . Applying the hypothesis together with Remark 46, we deduce the existence of a Borel section $\tau : G \rightarrow E^0 \subseteq E$ of the projection $E \rightarrow G$ such that the corresponding cocycle $c_\tau : G \times G \rightarrow A$ is bounded. As $[c_\tau] = x$, we conclude that x is bounded.

The fact that (3) implies (2) is trivial and obviously (3) implies (1), since $\pi_1(G)$ is a finitely generated abelian group. It remains to see that (2) implies (3). Let $A = \mathbb{Z}^n \times F$ be a finitely generated abelian group, F a finite group, and let $c : G \times G \rightarrow A$ be a 2-cocycle; it has components $c_i : G \times G \rightarrow \mathbb{Z}$, for $1 \leq i \leq n$ and a component c_F , taking values in F . The latter is obviously bounded. By assumption, the cocycles c_i are cohomologous to bounded cocycles d_i . Thus, the cocycle $G \times G \rightarrow A$ with components d_i and c_F is bounded and cohomologous to c , finishing the proof.

Lemma 48 *If G and H are two virtually connected Lie groups such that the maps $H^2_{Bb}(G, \mathbb{Z}) \rightarrow H^2_B(G, \mathbb{Z})$ and $H^2_{Bb}(H, \mathbb{Z}) \rightarrow H^2_B(H, \mathbb{Z})$ are onto then so is the map $H^2_{Bb}(G \times H, \mathbb{Z}) \rightarrow H^2_B(G \times H, \mathbb{Z})$.*

Proof Because $\pi_1(BG) = \pi_0(G)$ is finite, $H^1(BG, \mathbb{Z}) = 0$, and similarly $H^1(BH, \mathbb{Z}) = 0$. By the Künneth formula in singular cohomology, we conclude that the natural map $BG \sqcup BH \rightarrow BG \times BH = B(G \times H)$ induces an isomorphism ψ at the level of degree 2 cohomology with \mathbb{Z} coefficients. On the other hand, the inclusion $i_G : G \rightarrow G \times H$ induces a surjection i_G^* in Borel bounded cohomology (because there is a retraction $G \times H \rightarrow G$), and similarly for the inclusion i_H . The following diagram commutes because for each component of the vertical maps we have a natural commutative diagram:

$$\begin{array}{ccc}
 H_{Bb}^2(G \times H, \mathbb{Z}) & \longrightarrow & H_B^2(G \times H, \mathbb{Z}) = H^2(B(G \times H), \mathbb{Z}) \\
 \text{epi} \downarrow (i_G^*, i_H^*) & & \text{iso} \downarrow \psi \\
 H_{Bb}^2(G, \mathbb{Z}) \oplus H_{Bb}^2(H, \mathbb{Z}) & \xrightarrow{\text{epi}} & H^2(BG, \mathbb{Z}) \oplus H^2(BH, \mathbb{Z}).
 \end{array}$$

This shows that the map $H_{Bb}^2(G \times H, \mathbb{Z}) \rightarrow H_B^2(G \times H, \mathbb{Z})$ is onto as well.

Proposition 49 *Let G be a connected Lie group and let N be a closed connected normal subgroup of G . The natural map*

$$H^2(BG, \mathbb{R}) \rightarrow H^2(BN, \mathbb{R}),$$

induced by the inclusion $N \subset G$, is onto.

Proof There is a commutative diagram:

$$\begin{array}{ccc}
 H^2(BG, \mathbb{Z}) & \longrightarrow & H^2(BN, \mathbb{Z}) \\
 \cong \downarrow & & \cong \downarrow \\
 \text{Hom}(\pi_1(G), \mathbb{Z}) & \longrightarrow & \text{Hom}(\pi_1(N), \mathbb{Z}).
 \end{array}$$

Tensoring with \mathbb{R} , we see that the proof will be finished if we show that

$$\text{Hom}(\pi_1(G), \mathbb{R}) \rightarrow \text{Hom}(\pi_1(N), \mathbb{R})$$

is onto. But this is the case according to Remark 5.

4 Proof of the main results

4.1 Proof of some auxiliary results

Lemma 50 *Let G be a connected Lie group and \sqrt{G} its radical. The restriction map $H^*(BG, \mathbb{R}) \rightarrow H^*(B\sqrt{G}, \mathbb{R})$ is surjective.*

Proof According to Proposition 49, the map

$$H^2(BG, \mathbb{R}) \rightarrow H^2(B\sqrt{G}, \mathbb{R})$$

is surjective. Because \sqrt{G} is homotopy equivalent to a torus, the cohomology $H^*(B\sqrt{G}, \mathbb{R})$ is a polynomial algebra on 2-dimensional classes. Therefore, since the restriction map $H^*(BG, \mathbb{R}) \rightarrow H^*(B\sqrt{G}, \mathbb{R})$ is a map of algebras, it is surjective.

Lemma 51 *Let G be a connected Lie group and \sqrt{G} its radical. Then $H^*(BG, \mathbb{R})$ is generated as an algebra by $H^2(BG, \mathbb{R})$ together with the image of the inflation map $H^*(B(G/\sqrt{G}), \mathbb{R}) \rightarrow H^*(BG, \mathbb{R})$.*

Proof Let I^* denote the image of $H^*(B(G/\sqrt{G}), \mathbb{R}) \rightarrow H^*(BG, \mathbb{R})$. By Lemma 50, the restriction map $H^*(BG, \mathbb{R}) \rightarrow H^*(B\sqrt{G}, \mathbb{R})$ is surjective and therefore the fiber in the fibration $B\sqrt{G} \rightarrow BG \rightarrow BG/\sqrt{G}$ is totally non-homologous to zero relative to \mathbb{R} . We conclude from Borel's Theorem [4, Theorem 14.2]) that $H^*(BG, \mathbb{R})/\langle I^+ \rangle$ maps isomorphically onto $H^*(B\sqrt{G}, \mathbb{R})$, where $\langle I^+ \rangle$ stands for the ideal generated by the classes $I^+ < I^*$ of positive degree. Using the well-known fact that for a connected Lie group L , $H^*(BL, \mathbb{R})$ is a polynomial algebra on even dimensional generators (see [4, Theorem 18.1]) we now prove our claim on the generation of $H^*(BG, \mathbb{R})$ by checking it for generators in even degrees. Thus, assume that $x \in H^{2d}(BG, \mathbb{R})$ is a generator with $d > 1$. Then x maps to an element $y \in H^{2d}(B\sqrt{G}, \mathbb{R})$, which is a product of d 2-dimensional classes y_i . By choosing counter images x_i in $H^2(BG, \mathbb{R})$ for the y_i 's, their product will be an element $z \in H^{2d}(BG, \mathbb{R})$ with $x - z \in \langle I^+ \rangle$, say $x - z = \sum a_j b_j$ with $a_j \in I^+$ and $b_j \in H^{<2d}(BG, \mathbb{R})$. Thus $x = z + \sum a_j b_j$ lies in the subring generated by $H^2(BG, \mathbb{R})$ together with $\langle I^+ \rangle$, proving the theorem.

Lemma 52 *Let G be a connected Lie group. Assume that the radical \sqrt{G} of G is linear. There is a closed contractible subgroup $V < \sqrt{G}$ which is normal in G such that \sqrt{G}/V is a torus T and the covering space ξ induced from the universal cover $\tilde{L} \rightarrow G/\sqrt{G}$ via the canonical projection $G/V \rightarrow G/\sqrt{G}$ is of the form $\xi : T \times \tilde{L} \rightarrow G/V$.*

Proof Let H be the covering space of G induced from the universal cover \tilde{L} of G/\sqrt{G} . Then, according to (44) of Lemma 44 and Lemma 14, $H = \sqrt{G} \times \tilde{L}$, with \tilde{L} semi-simple. Let W be the linearizer of \tilde{L} ; it is discrete and central in \tilde{L} . Also W is central in H because W lies in the kernel of the adjoint representation of H . So we can form the quotient

$$Q := \sqrt{G} \times (\tilde{L}/W),$$

which is a linear Lie group by Theorem 2. According to Hochschild [22, XVIII. 4, Theorem 4.3], the group Q contains a contractible normal closed solvable subgroup X with Q/X linear reductive. Since $X < \sqrt{G}$ we can think of X as a subgroup of $H = \sqrt{G} \times \tilde{L}$, and this subgroup is normal, since W is central. Under the covering projection $H \rightarrow G$, this subgroup X maps isomorphically onto a closed normal subgroup $V < G$, which lies in the radical of G . Since Q/X is linear reductive, \sqrt{G}/X is a central torus T (see [22, XVIII. 4, Theorem 4.4]). Let ξ be the covering induced

from the universal cover $\tilde{L} \rightarrow G/\sqrt{G}$ via the canonical projection $G/V \rightarrow G/\sqrt{G}$. According to (44) of Lemma 44, the total space E of ξ sits in the short exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow \tilde{L} \rightarrow 1.$$

As \tilde{L} is simply-connected, and as T is the radical of E , the sequence splits (again by Lemma 14). Since T is central, we deduce that E is a direct product $E = T \times \tilde{L}$.

4.2 Proof of Theorem 1

4.2.1 (1) \Rightarrow (2) in Theorem 1

Proof We first prove that all classes in $H_B^2(G, \mathbb{Z})$ are bounded. Since \sqrt{G} is linear, Lemma 52 implies that G contains a normal contractible subgroup V , hence $G \rightarrow G/V$ is a homotopy equivalence, and thus $H_B^*(G/V, \mathbb{Z}) \rightarrow H_B^*(G, \mathbb{Z})$ an isomorphism by Lemma 33. We can therefore replace G by G/V . In the notation of Lemma 52, we have covering spaces $\mathbb{R}^n \times \tilde{L} \rightarrow T \times \tilde{L} \rightarrow G/V$. Note that $\mathbb{R}^n \times \tilde{L}$ is the universal cover of G/V . By Proposition 47, it suffices to show that the cocycle defining this universal cover can be chosen bounded. For this, we appeal to Lemma 45, with $X = \mathbb{R}^n \times \tilde{L}$, $Y = T \times \tilde{L}$ and $Z = G/V$. The map $p : X \rightarrow Y$ is the product of $\mathbb{R}^n \rightarrow T$ with $\text{Id} : \tilde{L} \rightarrow \tilde{L}$ and we conclude from Lemma 48 and Corollary 30 that the Borel cocycle associated with p can be chosen to be bounded. For $q : Y \rightarrow Z$ we observe that q is obtained by pulling back $\tilde{L} \rightarrow G/\sqrt{G}$ over G/V . According to (44) of Lemma 44, it suffices to prove that the universal cover $\tilde{L} \rightarrow G/\sqrt{G}$ has a Borel section σ such that the associated cocycle c_σ is bounded. But this case has already been dealt with in Corollary 36 and it follows that $H_{Bb}^2(G, \mathbb{Z}) \rightarrow H_B^2(G, \mathbb{Z})$ is surjective. To pass to higher degrees, we will make use of Lemma 51. Since for every d , $H^d(BG, \mathbb{Z})$ is a finitely generated abelian group, we see that the \mathbb{Z} -subalgebra $A^* \subseteq H^*(BG, \mathbb{R})$ generated by $H^2(BG, \mathbb{Z})$ together with the image J^* of the inflation map $H^*(B(G/\sqrt{G}), \mathbb{Z}) \rightarrow H^*(BG, \mathbb{Z})$ satisfies, in view of Lemma 51, $A^* \otimes \mathbb{R} = H^*(BG, \mathbb{R})$. Thus $A^d \subseteq H^d(BG, \mathbb{Z})$ has finite index for every d . To show that $H_{Bb}^d(G, \mathbb{Z}) \rightarrow H_B^d(G, \mathbb{Z})$ is onto it suffices therefore by Lemma 25 to show that the elements of A^d all lie in the image of $H_{Bb}^d(G, \mathbb{Z}) \rightarrow H_B^d(G, \mathbb{Z})$. From the definition of A^* it suffices therefore to show that:

- (a) the map $H_{Bb}^2(G, \mathbb{Z}) \rightarrow H_B^2(G, \mathbb{Z})$ is onto;
- (b) the map $H_{Bb}^*(G/\sqrt{G}, \mathbb{Z}) \rightarrow H_B^*(G/\sqrt{G}, \mathbb{Z})$ is onto.

Part (a) we just proved and part (b) follows from Corollary 36, since G/\sqrt{G} is semi-simple.

4.2.2 (2) \Rightarrow (3) in Theorem 1

Proof By Proposition 47 we conclude that the class defined by the universal cover is bounded. Let $A = \pi_1(G)$ be the fundamental group of G . We have a topological A -extension:

$$0 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where \tilde{G} denotes the universal cover of G . Let τ be a Borel section of the projection $\tilde{G} \rightarrow G$. By hypothesis the class $[c_\tau] \in H_B^2(G, A)$ is bounded. According to Lemma 39, there exists a Borel section σ of $\tilde{G} \rightarrow G$ such that c_σ is bounded. We may modify σ at the identity so that $\sigma(e) = e$. It is easy to check that the new cocycle c_σ associated to the modified section σ is still bounded. Proposition 42 implies that A is undistorted in \tilde{G} .

4.2.3 (3) \Rightarrow (1) in Theorem 1

Proof Suppose that \sqrt{G} is not linear. Then according to Proposition 22, the fundamental group of G is distorted in \tilde{G} , contradicting assumption (3).

The proof presented in 4.2.2 shows that if $H_B^2(G, \mathbb{Z})$ is bounded, then (3) in Theorem 1 holds. Using Proposition 47, we deduce that the equivalent conditions of Theorem 1 are also equivalent to the ones of the following Corollary.

Corollary 53 *Let G be a connected Lie group. The following are equivalent.*

1. *Each Borel cohomology class of G with \mathbb{Z} -coefficients can be represented by a Borel bounded cocycle.*
2. *Each Borel cohomology class of G of degree two with \mathbb{Z} -coefficients can be represented by a Borel bounded cocycle.*
3. *The class in $H_B^2(G, \pi_1(G))$ defined by the universal cover of G can be represented by a Borel bounded cocycle.*

5 Relationship with Gromov’s Theorem

For the proof of Theorem 1.1. we made use of Gromov’s Theorem on the boundedness of primary characteristic classes of flat G -bundles, where G is the Lie group of \mathbb{R} -points of a connected linear algebraic group over \mathbb{R} (cf. [17, Sect. 1.3, p. 23]). On the other hand, our Theorem 1 implies the following generalization of Gromov’s Theorem.

Theorem 54 *Let G be a virtually connected Lie group with linear radical. The image of the natural map*

$$H^*(BG, \mathbb{R}) \rightarrow H^*(BG^\delta, \mathbb{R})$$

consists of bounded classes.

Proof Using Proposition 31 and because $\sqrt{G} = \sqrt{G^0}$ it suffices to consider the case of a connected group G . Thus, By Theorem 1, $H_{Bb}^*(G, \mathbb{Z}) \rightarrow H_B^*(G, \mathbb{Z})$ is surjective. From the commutative diagram displayed in Sect. 3.1 we then infer that the image of $H^*(BG, \mathbb{R}) \rightarrow H^*(BG^\delta, \mathbb{R})$ is contained in the image of $H_b^*(G^\delta, \mathbb{R}) \rightarrow H^*(G^\delta, \mathbb{R})$.

Remark 55 The converse of the previous result does not hold; that is, from the boundedness of primary real characteristic classes for flat G -bundles on cannot conclude

that \sqrt{G} is linear. For instance, if $G = S^1 \times \mathbf{H}/\mathbb{Z}$ is the group discussed in Remark 12 then, because G is solvable with $\pi_1([G, G]) = 0$, Goldman's result [15] implies that $\phi : H^*(BG, \mathbb{R}) \rightarrow H^*(BG^\delta, \mathbb{R})$ is trivial in positive dimensions and, therefore, the image of ϕ consist for trivial reasons of bounded classes. But $\pi_1(\overline{[G, G]}) \neq 0$, which implies that $\sqrt{G} = G$ is non-linear.

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