CRASH-COURSE ON TOPOLOGICAL K-THEORY FOR C*-ALGEBRAS

I. L. CHATTERJI

This text is based on the following sources, that I recommend to any of those who want to learn the subject a little more seriously.

[1] **J.B. Conway**, *A Course in Functional Analysis.* (Basics concerning C^*-algebras.)
[2] **G. J. Murphy**, *C^*-algebras and operator theory.* (Basics concerning C^*-algebras, with some K-theory.)
[3] **M. Takesaki**, *Theory of Operators algebras I.* (Basics concerning C^*-algebras.)
[4] **J.L. Taylor**, *Banach Algebras and Topology.* (A nice introduction to topological K-theory.)
[5] **N. E. Wegge-Olsen**, *K-Theory and C^*-algebras.* (A "friendly approach" to topological K-theory, with a lot of details worked out carefully. Basics of C*-algebras are assumed.)
[6] **A. Valette**, *Introduction to the Baum-Connes conjecture.* (Chapter 3 gives a short introduction to topological K-theory. Basics of C*-algebras are assumed.)

1. Introduction

The aim of what follows is to give a crash-course in K-theory of C*-algebras. This theory defines a collection of functors $\{K_n\}_n$ from the category of C*-algebras to the category of abelian groups, satisfying the Eilenberg-MacLane axioms for a homology theory[1]. A nice feature of this theory is the Bott periodicity, which implies that those functors are actually only two. The subject covered in the following sections does work as well for any Banach algebra. The point of considering only C*-algebras is first because my favourite mathematical objects happen to be C*-algebras[2], and secondly because this is the only generality that one needs to have to get the link with the K-theory of topological

Date: November 7, 2002.

[1]This sentence is abstract nonsense, and understanding it is not required for this text. The same applies to the rest of the Introduction.

[2]The reduced C*-algebra of torsion-free groups.

spaces, which turns out to be the K-theory for a particular class of C*-algebras, namely those of the form $C_0(X)$ – the continuous functions over a locally compact Hausdorff topological space X, vanishing at infinity.

Warning: This text contains mistakes. Please open your eyes and report any mistake.

2. C*-algebras

In this section we shall state some basic definitions related to C*-algebras.

Definition 2.1. A *Banach space* A over $\mathbf{C}$ is a complete normed space over $\mathbf{C}$. If furthermore A is an algebra over $\mathbf{C}$ and its multiplication satisfies the inequality $\|xy\| \leq \|x\|\|y\|$ for each $x, y \in A$, then A is called a *Banach algebra*, and an *involutive Banach algebra* if it is endowed with a map $A \to A$, $x \mapsto x^*$ satisfying the following properties:

$$\begin{aligned} &x^{**} = x \\ &(x+y)^* = x^* + y^* \\ &(\alpha x)^* = \bar{\alpha} x^* \\ &(xy)^* = y^* x^* \\ &\|x^*\| = \|x\| \text{ for each } x, y \in A,\ \alpha \in \mathbf{C}. \end{aligned}$$

An involutive Banach algebra that satisfies the equality $\|x^*x\| = \|x\|^2$ (C*-equation) is called a *C*-algebra*.

Example 2.2. 1) Let X be a locally compact Hausdorff topological space; we write $C_0(X)$ for the algebra of functions vanishing at infinity. We recall that $C_0(X)$ consits of continuous functions $f : X \to \mathbf{C}$ such that for each $\epsilon > 0$ there exists a compact K_ϵ in X such that $f(x) < \epsilon$ for all x outside K_ϵ. Then $C_0(X)$, with pointwise addition and multiplication, is an abelian C*-algebra; the involution is given by $f^*(x) = \overline{f(x)}$ for each $f \in C_0(X), x \in X$ and the norm is given by $\|f\| = sup\{|f(x)|, x \in X\}$. The space X is compact if and only if this algebra has a unit (i.e. a $1 \in C_0(X)$ such that $1f = f1 = f$ for each $f \in C_0(X)$). In this case $C_0(X) = C(X)$ the continuous functions from X to $\mathbf{C}$.

2) For n a positive integer, $M_n(\mathbf{C})$ the algebra of $n \times n$ matrices with coefficients in $\mathbf{C}$ is a C*-algebra for the norm

$$\|M\| = \sup\{Mx | x \text{ is a unit vector in } \mathbf{C}^n\}$$

($M = (m_{ij}) \in M_n(\mathbf{C})$), and involution given by $(m_{ij})^* = (\overline{m}_{ji})$.

Definition 2.3. A *pre-Hilbert space* is a complex vector space $\mathcal{H}$, endowed with a *scalar product*, namely a map

$$\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{C}$$

satisfying, for all $v, v', w \in \mathcal{H}$ and $\lambda \in \mathbf{C}$:

$$\begin{aligned}
\langle v, w\rangle &= \overline{\langle w, v\rangle} \\
\langle v + v', w\rangle &= \langle v, w\rangle + \langle v', w\rangle \\
\langle \lambda v, w\rangle &= \lambda \langle v, w\rangle \\
\langle v, v\rangle &\geq 0 \text{ with equality if and only if } v = 0
\end{aligned}$$

A *Hilbert space* is a complete pre-Hilbert space (with respect to the topology coming from the scalar product). We get a norm on $\mathcal{H}$ by setting $\|v\| = \sqrt{\langle v, v\rangle}$ for $v \in \mathcal{H}$.

Example 2.4. For any integer n, $\mathbf{C}^n$ is a Hilbert space (of finite dimension n). For $\mathbf{S}$ any set, consider

$$\ell^2(\mathbf{S}) = \{f : \mathbf{S} \to \mathbf{C} \text{ such that } \sum_{n\in\mathbf{S}} |f(n)|^2 < \infty\},$$

one defines a scalar product as follows (for $f, g \in \ell^2(\mathbf{S})$)

$$\langle f, g\rangle = \sum_{n\in\mathbf{S}} f(n)\overline{g(n)}.$$

Denote by δ_s the element of $\ell^2(\mathbf{S})$ taking value 1 in $s \in \mathbf{S}$ and 0 otherwise (we call the *Dirac functions*), $\{\delta_s\}_{s\in\mathbf{S}}$ is called *Hilbert basis*, as any element of $\ell^2(\mathbf{S})$ can be expressed as an infinite linear combination of the δ_s's, with square summable coefficients. Any infinite dimensional Hilbert space with countable basis is isomorphic to $\ell^2(\mathbf{N})$, and any Hilbert space admits an orthonormal basis (but not necessarily countable). We shall only consider separable Hilbert spaces.

Remark 2.5. Let $\mathcal{H}$ be a Hilbert space, the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear maps from $\mathcal{H}$ to itself is a C*-algebra for the operator norm:

$$\|P\| = \sup\{Px \text{ such that } x \text{ is a unit vector in } \mathcal{H}\}$$

($P \in \mathcal{B}(\mathcal{H})$). Any such operator has a unique *adjoint*, namely an element $P^* \in \mathcal{B}(\mathcal{H})$ such that for any $v, w \in \mathcal{H}$

$$\langle Pv, w\rangle = \langle v, P^*w\rangle$$

The involution is the map sending an operator to its adjoint. This C*-algebra is non abelian as soon as the dimension of $\mathcal{H}$ exceeds one.

More generally, any closed sub-*-algebra A of $\mathcal{B}(\mathcal{H})$ (that is any closed sub-algebra of $\mathcal{B}(\mathcal{H})$ which is invariant under taking the adjoint) is a C*-algebra. Conversely, any C*-algebra can be seen as a closed sub-*-algebra of bounded operators on a Hilbert space, using the GNS (Gelfand-Naimark-Segal) construction, see [3] or [1].

Example 2.6. 1) As a concrete illustration of the previous remark, one considers a discrete group Γ, and the Hilbert space

$$\ell^2(\Gamma) = \{f : \Gamma \to \mathbf{C} \text{ such that } \sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty\}.$$

We mention the following important subalgebras of $\mathcal{B}(\ell^2(\Gamma))$:

The *reduced C*-algebra of* Γ, denoted by $C_r^*(\Gamma)$ which is the closure (for the operator norm) of the *-algebra generated operators of the form $L_\gamma(f)(\mu) = f(\gamma^{-1}\mu)$, for $f \in \ell^2(\Gamma), \gamma, \mu \in \Gamma$. This amounts to embedding the group ring $\mathbf{C}\Gamma$ in $\mathcal{B}(\ell^2(\Gamma))$ by letting elements act by left convolution, and then close this embedding with respect to the operator norm on $\mathcal{B}(\ell^2(\Gamma))$.

The *von Neumann algebra of* Γ, denoted by $\mathcal{N}(\Gamma)$ which consists of all elements $P \in \mathcal{B}(\ell^2(\Gamma))$ commuting with $\{L_\gamma | \gamma \in \Gamma\}$.

2) On the other hand, given a discrete group Γ we might want to look at $\ell^1(\Gamma)$ with the ℓ^1 norm ($\|f\| = \sum_{\gamma\in\Gamma} |f(\gamma)|$ for each $f \in \ell^1(\Gamma)$). This is an involutive Banach algebra but not a C*-algebra. Again the involution can be expressed as $f^*(\gamma) = \overline{f(\gamma^{-1})}$, but the C*-equation is not fulfilled by $f = e + ix$ in case $x^2 = e$ (for e being the neutral element of Γ), or by $f = e + x - x^2$ in case $x \neq x^2 \neq e$.

3) Example 1) extends to a locally compact topological group G as follows: take μ a Haar measure on G, then look at $C_c(G)$ the continuous functions with compact support as acting by left convolution on the Hilbert space of square integrable functions $L^2(G, \mu)$. In this way $C_c(G)$ embeds in $\mathcal{B}(L^2(G, \mu))$, and closing it with respect to the operator norm yields the reduced C*-algebra of G.

Remark 2.7. In the definition of C*-algebra we did not require the existence of a unit, but it can be added in the following way: We consider the set $A_I = \{(a, \lambda) | a \in A, \lambda \in \mathbf{C}\}$ with operations given as follows:

$$(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu) \text{ for all } a, b \in A,\ \lambda, \mu \in \mathbf{C}$$
$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu) \text{ for all } a, b \in A,\ \lambda, \mu \in \mathbf{C}$$
$$(a, \lambda)^* = (a^*, \bar{\lambda})$$

Then the unit is $(0,1)$ and the norm is given by $\|(a,\lambda)\| = sup\{\|(xy + \lambda y\|, \|y\| = 1\}$, that is, the operator norm of A_I acting on A. (The ℓ^1 norm given by $\|(a,\lambda)\| = \|a\| + \|\lambda\|$ would turn A_I into a Banach algebra which will not necessarily be a C*-algebra, see previous example.)

Any homomorphism $\varphi : A \to B$ between C*-algebras determines a unital homomporphism $\varphi_I : A_I \to B_I$ by $\varphi_I(a + \lambda) = \varphi(a) + \lambda$, for each $a \in A, \lambda \in \mathbf{C}\}$.

Example 2.8. Let $A = C_0(X)$ for X a locally compact non compact topological space, then A_I corresponds to $C(X^+)$ (continuous functions on X^+), where X^+ is the one point compactification of X and $C_0(X) \subset C(X^+)$ is the closed ideal of functions vanishing at the added point.

Definition 2.9. For a C*-algebra A with unit let us denote by $G(A)$ the set of invertible elements of A. It is a group under multiplication, and even a topological group (for the topology induced by the one of A), open in A. We will write $G^0(A)$ for the connected component of the unit in $G(A)$.

Now let a be an element of A, we define the exponential of a by the absolutely convergent serie

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}.$$

It is an element of $G(A)$, whose inverse is given by e^{-a}.

Remark 2.10. If two elements a and b of A commute, then $e^{a+b} = e^a e^b$. Let us consider $a \mapsto e^{2\pi i a} = \exp(a)$, that maps the abelian additive group $(A, +)$ to the multiplicative (possibly non abelian) group $(G(A), \cdot)$. This is in general not a group homomorphism.

Proposition 2.11. *Let A be a C*-algebra with unit, written 1. Then*

$$G^0(A) = < \exp(A) >,$$

where $G^0(A)$ is the connected component of the unit in $G(A)$, and $< \exp(A) >$ is the multiplicative subgroup of $G(A)$ generated by elements of the form e^a for $a \in A$.

Proof. To begin with, $< \exp(A) >$ is non empty since $1 = e^0$, which lies in $< \exp(A) >$.

Then $< \exp(A) >$ is arc-wise connected since each element x of $< \exp(A) >$ can be expressed as $x = e^{a_1} \ldots e^{a_n}$, where $a_1 \ldots a_n \in A$ and thus is connected to the identity by the arc $x(t) = e^{ta_1} \ldots e^{ta_n}$, where $t \in [0,1]$.

Now $< \exp(A) >$ is open in $G(A)$, because for an $a \in A$ satisfying $\|1 - a\| < 1$ the convergent serie $-\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n$ gives a logarithm for a and thus an open neighbourhood V of 1 in $\exp(A)$. The multiplication by any element of $G(A)$ being a homeomorphism, for each $b \in <\exp(A)>$, Vb will be an open neighbourhood of b in $<\exp(A)>$.

Finally $<\exp(A)>$ is closed in $G(A)$ for it is an open subgroup of $G(A)$. $\square$

3. The functor K_1

For this section, A is a C*-algebra, with unit (unless specified).

Definition 3.1. For each $n \in \mathbf{N}$ we will write $M_n(A)$ for the C*-algebra of $n \times n$ matrices with coefficients in A. The norm is the operator norm that we get by considering $M_n(A)$ as an algebra of operators on $\oplus_n A$ (for $a = (a_1, \dots, a_n) \in \oplus_n A$, one defines $\|a\| = \sqrt{\|a_1\|^2 + \cdots + \|a_n\|^2}$). We will write $GL_n(A)$ for the multiplicative group of invertible elements of $M_n(A)$ (this is $G(M_n(A))$ following the notations given in the previous section). Let $m, n \in \mathbf{N}$ and $a \in M_n(A)$, $b \in M_m(A)$, then $a \oplus b$ is defined as the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $M_{n+m}(A)$, and is called *trivial extension* of a any matrix of the form $a \oplus I \in M_k(A)$ for $k > n$, and I the identity of $M_{k-n}(A)$. In this way we can embed $GL_n(A)$ in $GL_{n+1}(A)$ and define

$$GL(A) = \lim_{\rightarrow} GL_n(A)$$

Remark 3.2. We recall that a direct system of groups is a family $(C_\alpha, f_{\alpha\beta})_{\alpha \in \phi}$ indexed by a partially ordered filtering set ϕ, where $f_{\alpha\beta} : C_\alpha \to C_\beta$ are group homomorphisms satisfying the conditions that $f_{\alpha\alpha}$ is the identity and $f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma}$ when $\alpha \leq \beta$ and for each $\alpha \leq \beta \leq \gamma$. For such a direct system, the direct limit, denoted by $\lim_{\rightarrow} C_\alpha$ is a family (L, f_α) again indexed by ϕ, where for each $\alpha \in \phi$, one has $f_\alpha : C_\alpha \to L$ satisfies $f_\beta f_{\alpha\beta} = f_\alpha$ and is universal in the sense that if (D, g_α) is another object satisfying the two previous conditions, then one can find a $\varphi : D \to L$ such that $g_\alpha \varphi = f_\alpha$ for each $\alpha \in \phi$.

In our case the family $(GL_n(A), \cdot \oplus I)_{n \in \mathbf{N}}$ is clearly a direct system of groups, and its direct limit is the object we are interested in. We might want to think about this direct limit as the infinite dimensional matrices of the form $\begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}$, where a belongs to $GL_n(A)$ for an $n \in \mathbf{N}$ and I is an infinite dimensional identity.

Proposition 3.3. *Let $n \in \mathbf{N}$ and $a, b \in Gl_n(A)$, then $ab \oplus I$ and $ba \oplus I$ are in the same connected component of $GL_{2n}(A)$.*

Proof. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is connected to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by the arc $t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ in $GL_2(\mathbf{C})$, and the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is connected to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by the arc $t \mapsto \begin{pmatrix} 0 & e^{it} \\ 1 & 0 \end{pmatrix}$ in $GL_2(\mathbf{C})$, and this fact extended to the $n \times n$ matrices shows that shuffeling rows and colums of a matrix of $GL_n(\mathbf{C})$ will not change connected component. Hence $ab \oplus I = (a \oplus I)(b \oplus I)$ is in the same connected component to $(a \oplus I)(I \oplus b) = a \oplus b$, which is connected to $b \oplus a$ and $I \oplus ba$ by the same argument. □

Definition 3.4. For $n \geq 1$, define the *n-th K-group* of A by

$$K_n(A) = \pi_{n-1}(A).$$

Remark 3.5. Proposition 3.3 shows that $K_1(A)$ is an abelian group. We shall define $K_0(A)$ in the next section, and we shall see that $K_n(A) = K_{n+2}(A)$ for any $n \geq 0$ (this is Bott periodicity), so that there are actually only two of those functors.

Proposition 3.6. *The group $K_1(\mathbf{C})$ is trivial.*

Proof provided by G. Valette. It is enough to show that for any n, $GL_n(\mathbf{C})$ is connected. To do this, take a matrix $A \in GL_n(\mathbf{C})$ and consider $A_z = (1-z)I + zA$. There are at most n points in $\mathbf{C}$ where A_z is not invertible (those are the zeros of the caracteristic polynomial of A), so that we conclude by connectedness of $\mathbf{C}$ minus n points. □

Until now we assumed and widely used the fact that our C*-algebra A had a unit, but we'll now define K-groups for any C*-algebra A.

Definition 3.7. The *n-th K-group* of A is given by the kernel of $\varphi^* : K_1(A_I) \to K_1(\mathbf{C})$, and again denoted by $K_1(A)$, where φ^* is induced by $\varphi : A_I \to \mathbf{C}$.

Remarks 3.8. Since we just saw that $K_1(\mathbf{C})$ is trivial, we now have that $K_1(A) = K_1(A_I)$. Furthermore, $A \to A_I$ being a functor from the category of C*-algebras to the category of unital C*-algebras, it means that we just extended K_1 to a functor from the category of C*-algebras to the category of abelian groups.

Proposition 3.9 (Weak exactness)**.** *Let $\mathcal{J} \subset A$ be a closed ideal. The exact sequence $0 \to \mathcal{J} \to A \to A/\mathcal{J} \to 0$ induces an exact sequence $K_1(\mathcal{J}) \to K_1(A) \to K_1(A/\mathcal{J})$.*

Proof. First we extend $i : \mathcal{J} \to A$ and $\pi : A \to A/\mathcal{J}$ to $i : \mathcal{J}_I \to A_I$ and $\pi : A_I \to (A/\mathcal{J})_I$, so that $\pi \cdot i$ sends $\mathcal{J}_I$ on $\mathbf{C}$, and $K_1(\mathbf{C})$ being trivial, we conclude that $\pi^* \cdot i^* = 0$, that is to say $\mathrm{im}(i^*) \subset \ker(\pi^*)$.

For the other inclusion, notice that an element u in the kernel of π^* can be written as $u = [a \oplus I]$, where a is an element of $Gl_n(A_I)$ such that $\pi(a)$ belongs to $GL_n^0(A_I/\mathcal{J})$, which means that $\pi(a) = e^{a_1} \dots e^{a_m}$, with the a_i's in $M_n(A_I/\mathcal{J})$. Setting $b = e^{-b_m} \dots e^{-b_1} a$ where the b_i's are in $M_n(A_I)$ pre-images of the a_i's, we have that b is an element of $GL_n(A_I)$, that $[b \oplus I] = u$ (since b differs from a by elements belonging to $GL_n^0(A_I)$), and that $b \in GL_n(\mathcal{J}_I)$ (since $\pi(b) = I$), so that $u \in \mathrm{im}(i^*)$. □

4. The functor K_0

Definition 4.1. A *semi-group* is a set S endowed with an associative law $S \times S \to S$, we call it *abelian* whenever this law is commutative. Let S be an abelian semi-group, then there exists an abelian group $U(S)$ called the *universal group associated* to S and a map $\mu : S \to U(S)$ such that for each group G and map $\varphi : S \to G$ there is a unique homomorphism $\tilde{\varphi} : U(S) \to G$ satisfying $\tilde{\varphi} \circ \mu = \varphi$.

Remark 4.2. For a given semi-group S, the group $U(S)$ can be canonically built as follows: Consider $S \times S$ with the equivalence relation $(x, y) \sim (u, v)$ if it exists an element $r \in S$ such that $x+v+r = y+u+r$ and define $U(S) = S \times S/\sim$. Then (x, x) will be the neutral element and (y, x) the inverse of (x, y). The map $\mu : S \to U(S)$ is given by $x \mapsto [(x + r, r)]$, and for a group G and map $\varphi : S \to G$, the homomorphism $\tilde{\varphi} : U(S) \to G$ will be $\tilde{\varphi}(x, y) = \varphi(x) - \varphi(y)$.

Definition 4.3. For each $n \in \mathbf{N}$ let $P_n(A)$ be the set of idempotent matrices of $M_n(A)$. Let us consider $\cup_{n\in\mathbf{N}} P_n(A)$ with the following equivalence relation: $p \in P_n(A)$ and $q \in P_m(A)$ (for $m, n \in \mathbf{N}$) are equivalent ($p \sim q$) if one can find $k \in \mathbf{N}$, $k \geq n, m$ and $u \in Gl_k(A)$ such that $p \oplus 0_{k-n} = u(q \oplus 0_{k-m})u^{-1}$ (the element $p \oplus 0_{k-n}$ is called *trivial extension of* p, and this equivalence relation means that we require p and q to be similar up to trivial extensions). Now $\cup_{n\in\mathbf{N}} P_n(A)/\sim$ is an abelian semi-group with the direct sum $\oplus$ (as previously defined) as associative law, and we define $K_0(A)$ as its associated universal group.

Remark 4.4. Straight from the construction of $K_0(A)$ and from the previous remark about universal group of a semi-group, by taking $G = K_0(A)$ and $\varphi = \mu$ we see that each element in $K_0(A)$ is written as a difference of two classes of idempotents $[p] - [q]$, for some $p \in P_n(A)$ and $q \in P_m(A)$. Which means that such two idempotents define the same element in $K_0(A)$ if, and only if one can find a third idempotent r such that $p \oplus r \sim q \oplus r$.

Furthermore, since $[p] - [q] = [p] - [q] + [q + I_k - q] - [I_k] = [p] - [q] + [q] + [I_k - q] - [I_k] = [p] + [I_k - q] - [I_k] = [p \oplus (I_k - q)] - [I_k]$ if I_k denotes the $k \times k$ identity matrix, we have that any element in $K_0(A)$ can be written as written as $[p] - [I_k]$, for some $p \in P_n(A)$ and $k \in \mathbf{Z}$.

At this point we are able to compute $K_0(\mathbf{C})$:

Proposition 4.5. *We have that* $K_0(\mathbf{C}) = \mathbf{Z}$.

Proof. The map

$$\begin{array}{ccc} \mathbf{Z} & \to & K_0(\mathbf{C}) \\ n & \mapsto & \mathrm{sign}(n)([I_{|n|}] - [0]) \end{array}$$

is injective, and since each idempotent matrix $p \in M_n(\mathbf{C})$ is similar to a diagonal idempotent matrix with coefficients in $\mathbf{C}$, that is a matrix of the form $I_k \oplus 0_{n-k}$ for $k \leq n$ we get the surjectivity. □

Again until now we assumed and widely used the fact that our C*-algebra A had a unit, but we will now define $K_0(A)$ for any C*-algebra A.

Definition 4.6. The group $K_0(A)$ is given by the kernel of $\varphi_* : K_0(A_I) \to K_0(\mathbf{C}) \simeq \mathbf{Z}$, where the map φ_* is induced by $\varphi : A_I \to \mathbf{C}$ (whose kernel is A).

The following lemma shows that the topology of A enters somehow automatically in $K_0(A)$.

Lemma 4.7. *Let A be a unital C^*-algebra (the lemma also works for a unital Banach algebra) and e, f be two idempotents in $M_n(A)$ such that $\|e - f\| < \dfrac{1}{\|2e - 1\|}$. Then there exists an element $z \in GL_n(A)$ such that $f = z^{-1}ez$. In particular, e and f define the same class in $K_0(A)$.*

Proof. Set

$$z = \frac{1}{2}((2e - 1)(2f - 1) + 1).$$

Then $1 - z = (2e - 1)(e - f)$, so that $\|1 - z\| < 1$ and therefore z is invertible. Moreover, $ez = zf (= ef)$. □

Proposition 4.8 (Weak exactness)**.** *Let $\mathcal{J} \subset A$ be a closed ideal. The exact sequence $0 \to \mathcal{J} \to A \to A/\mathcal{J} \to 0$ induces an exact sequence $K_0(\mathcal{J}) \to K_0(A) \to K_0(A/\mathcal{J})$.*

In order to proceed with the proof we need the following lemma:

Lemma 4.9. *Let A have a unit. Given an $a \in G(A)$, we have that $a \in G^0(A)$ if and only if for each surjective homomorphism $B \to A$ (where B is a Banach algebra), a is the image of an invertible element of B.*

Proof of the lemma. If $a \in G^0(A)$ then $a = e^{a_1} \dots e^{a_n}$, with $b_1 \dots b_n$ pre-images of the a_i's. Then $b = e^{b_1} \dots e^{b_n}$ is invertible and pre-image of a.

Conversely, consider $B = \{f \in C([0,1], A) \mid f(0) = \lambda \cdot 1 \text{ for a } \lambda \in \mathbf{C}\}$ with the supremum norm. Then $f \mapsto f(1)$ is surjective (for $f(t) = (1-t) \cdot 1 + ta$ is pre-image of an $a \in A$), and if an f pre-image of a is invertible, that means that $f(t) \in G(A)$ for each $t \in [0,1]$ and thus $f(t)_{t \in [0,1]}$ is a path from a to 1 in $G(A)$. □

Proof of the proposition. Again we first extend $i : \mathcal{J} \to A$ and $\pi : A \to A/\mathcal{J}$ to $i : \mathcal{J}_I \to A_I$ and $\pi : A_I \to (A/\mathcal{J})_I$, so that $\pi \cdot i$ maps $\mathcal{J}_I$ on $\mathbf{C}$, which means that the induced map $\pi^* \cdot i^* : K_0(\mathcal{J}_I) \to K_0(A_I/\mathcal{J})$ extends the map $K_0(\mathcal{J}_I) \to K_0(\mathbf{C})$, whose kernel is $K_0(\mathcal{J})$ (by definition).

Conversely, given $c \in K_0(A)$, $c = [p] - [I_n]$ with $\pi^*(c) = 0$ means that p and I_n are conjugate in $K_0(A/\mathcal{J}) \subset K_0(A_I/\mathcal{J})$ (up to trivial extensions). The conjugation can be done trough an element of $GL^0_m(A_I)$ provided $m \in \mathbf{N}$ big enough (for $u \in GL_k(A/\mathcal{J})$, $u \oplus u^{-1} \in GL^0_{2k}(A/\mathcal{J})$ and $\pi^*(p) = uqu^{-1}$ implies $\pi^*(p) \oplus 0 = (u \oplus u^{-1})(q \oplus 0)(u \oplus u^{-1})^{-1}$), so by the previous lemma we can lift this element to an element v in $GL_k(A_I)$, and vpv^{-1} is $[I_n]$ modulo $\mathcal{J}$, which means that $[p]$ (and thus c) is in the image of i^*. □

5. Matching K_0 and K_1.

In this section we will see how the two functors K_0 and K_1 match together.

Proposition 5.1 (long exact sequence)**.** *Let $\mathcal{J} \subset A$ be a closed ideal, one has the following exact sequence:*

$$K_1(\mathcal{J}) \to K_1(A) \to K_1(A/\mathcal{J}) \to K_0(\mathcal{J}) \to K_0(A) \to K_0(A/\mathcal{J}).$$

We will not give the proof but just explain roughly how it works, for the proof see [4] or [5].

We only need to build a homomorphism $\delta^* : K_1(A/\mathcal{J}) \to K_0(\mathcal{J})$, and this is done as follows; given $a \in Gl_k(A/\mathcal{J})$, we know that $a \oplus a^{-1} \in G^0(M_{2k}(A/\mathcal{J}))$ has a pre-image $u \in Gl_{2k}(A)$. Let $p = u(I_k \oplus 0_k)u^{-1} \in P_{2k}(\mathcal{J}_I)$ (see last proof of the previous section), we define $\delta^*([a]) = [p] - [I_k]$. One now has to check that this is well defined (we chose a

and u), that $\delta^*([a])$ belongs to $K_0(\mathcal{J})$, that this is a homomorphism and the exactitude in $K_1(A/\mathcal{J})$ and $K_0(\mathcal{J})$.

Definition 5.2. The *cone* on a C*-algebra A is the set $CA = \{f \in C([0,1], A) \mid f(0) = 0\}$. It is a C*-algebra (for point-wise operations and supremum norm) while the *suspension* of A is the set $SA = \{f \in C([0,1], A) \mid f(0) = f(1) = 0\}$, which is again a C*-algebra for point-wise operations and supremum norm.

Proposition 5.3. *There is a natural isomorphism between* $K_1(A)$ *and* $K_0(SA)$.

Proof. The C*-algebra SA being a closed ideal of CA and the map $f \mapsto f(1)$ a surjective homomorphism from $CA \to A$ whose kernel is SA, that enables us to identify A with CA/SA. Using the long exact sequence we have that

$$K_1(SA) \to K_1(CA) \to K_1(A) \to K_0(SA) \to K_0(CA) \to K_0(A).$$

Then for each $f \in CA$, setting $\varphi_s f(t) = f(st)$ we get $s \mapsto \varphi_s$ a continuous path in $\mathrm{End}(CA)$ from 0 to 1 (that is, $\varphi_1 f = f$ and $\varphi_0 f = 0$ which extends to CA_I as usual by $\tilde{\varphi}_s(f+\lambda) = \varphi_s(f+\lambda)$), and that allow us to deform any matrix of $\cup_n P_n(CA_I)$ and $\cup_n Gl_n(CA_I)$ into a scalar matrix, which brings us back to the case of $\mathbf{C}$ and shows that $K_1(CA) = K_1(CA_I) = K_1(\mathbf{C}) = \mathbf{0}$ while $K_0(CA) = \ker(K_0(CA_I) \to K_0(\mathbf{C})) = \ker(\mathbf{Z} \to \mathbf{Z}) = \mathbf{0}$ (for the K_0 case, it is just Lemma 4.7). So we end up with the following exact sequence of groups:

$$0 \to K_1(A) \to K_0(SA) \to 0$$

which gives the seeked isomorphism. □

Remark 5.4. We recall that a C*-algebra A is *contractible* if there exists a path in $\mathrm{End}(A)$ connecting the zero map to the identity map. Notice that $\mathbf{C}$ is not contractible as a C*-algebra, but that SA (although in general not contractible) will be contractible if A is contractilble. The cone CA is always contractible (see proof above). More generally, the C*-algebra of continuous functions on a topologically contractible space X with target A, vanishing at one point is always contractible.

Proposition 5.5. *There is a natural isomorphism between* $K_0(A)$ *and* $K_1(SA)$.

The fairly long proof of this proposition will not be given here, it is done either in [4] or in [5]. However, we will roughly talk about an important map, namely the Bott map $\beta_A : K_0(A) \to K_1(SA)$ defined as follows:

For a given idempotent $p \in M_n(A)$ define

$$\begin{array}{rcl} f_p : [0,1] & \to & GL_n(A_I) \\ t & \mapsto & e^{2\pi i t p} \end{array}$$

Since $e^{2\pi i t p} = I + (e^{2\pi i t p} - 1)p$, we have that $f_p(0) = f_p(1) = I$, so that $f_p \in M_n((SA)_I)$, and $f_p(t)f_p(1-t) = e^{2\pi i p} = I$ implies that actually $f_p \in GL_n((SA)_I)$, so now we define:

$$\begin{array}{rcl} \beta_A : K_0(A) & \to & K_1(SA) \\ [p] - [q] & \mapsto & [f_p f_q^*], \end{array}$$

which is called the *Bott map.* It is the natural isomorphism between $K_0(A)$ and $K_1(SA)$.

This result, combined with the previous one, drives us straight to the central theorem of K-theory:

Theorem 5.6 (Bott periodicity)**.** *There is a natural isomorphism*

$$K_n(A) \simeq K_{n+2}(A)$$

for each $n \geq 0$.

Proof. Combining Proposition 5.3 and 5.5 gives $K_i(A) \simeq K_i(S^2A)$ for $i = 0, 1$ (where $S^2A = SSA$). Since for $n \geq 1$ $\pi_n(GL(A)) = \pi_{n-1}(SGL(A)) = \pi_{n-1}(GL(SA))$, it implies that $K_n(SA) = K_{n+1}(A)$. For $n = 0$, just use Proposition 5.5 once again. □

We reach an important corollary:

Corollary 5.7. *Let $I \subset A$ be a closed ideal. There is a repeating long exact sequence*

$$\cdots \to K_1(A) \to K_1(A/\mathcal{J}) \to K_0(\mathcal{J}) \to K_0(A) \to K_0(A/\mathcal{J}) \to K_1(\mathcal{J}) \to \dots$$

Remark 5.8. For a given locally compact topological space X, setting $K^p(X) = K_0(C_0(X))$ for p even, and $K^p(X) = K_1(C_0(X))$ for p odd gives the K-theory of topological spaces.

6. Further bibliography

We mention the following litterature dealing with topological K-theory for C^* or Banach algebras. The list is non exhaustive.

- M. Rørdam, F. Larsen, N. Laustsen. *An introduction to K-theory for C^*-algebras.* London Mathematical Society Student Texts, 49. Cambridge University Press, Cambridge, 2000. xii+242 pp. ISBN: 0-521-78334-8; 0-521-78944-3.

- P. A. Fillmore. *A user's guide to operator algebras.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996. xiv+223 pp. ISBN 0-471-31135-9

References

[1] J.B. Conway. *A Course in Functional Analysis.* Second edition. Graduate Texts in Mathematics, **96**. Springer-Verlag, New York, 1990. xvi+399 pp. ISBN: 0-387-97245-5

[2] G. J. Murphy. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990. x+286 pp. ISBN 0-12-511360-9

[3] M. Takesaki. *Theory of Operators algebras I.* Springer-Verlag, New York-Heidelberg, 1979. vii+415 pp. ISBN: 0-387-90391-7

[4] J.L. Taylor. *Banach Algebras and Topology* Algebras in Analysis. Proceedings of the London Mathematical Society and the University of Birmingham, edited by E. Williamson.

[5] N. E.. Wegge-Olsen *K-Theory and C*-algebras. A friendly approach.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. xii+370 pp. ISBN 0-19-859694-4

[6] A. Valette. *Introduction to the Baum-Connes conjecture.* From notes taken by I. Chatterji. With an Appendix by G. Mislin. ETH Zürich. Birkhäuser Verlag, Basel, 2002. x+104 pp. ISBN 3-7643-6706-7

E-mail address: `indira@math.cornell.edu`