

## Traces and reduced group $C^*$ -algebras

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ABSTRACT. We study the universal (Hattori-Stallings) trace on the K-theory of Banach algebras containing the complex group ring. As a result, we prove that for a group satisfying the Baum-Connes conjecture, finitely generated projectives over the reduced group  $C^*$ -algebra satisfy a condition reminiscent of the Bass conjecture. An immediate consequence is that in the torsion-free case, the change of ring map from the reduced group  $C^*$ -algebra to the von Neumann algebra of the group induces the zero map at the level of reduced K-theory.

### Introduction

A topological version of the Hattori-Stallings trace, or universal trace for the reduced  $C^*$ -algebra of a group  $G$  allows to formulate an analogue of the Bass conjecture [B]. Our main theorem shows that this analogue holds for many groups, namely we prove the following.

**THEOREM A.** *Suppose that a group  $G$  satisfies the Baum-Connes conjecture. Then the Hattori-Stallings trace*

$$HS^{C_r^*} : K_0(C_r^*G) \longrightarrow HH_0^{top}(C_r^*G)$$

*maps into the  $\mathbb{C}$ -vector space spanned by the images of the elements of finite order of  $G$ .*

The basics to understand the above result will be given in Section 1, and the proof will be given in Section 3. As an application, we obtain informations on the change of ring map from the reduced group  $C^*$ -algebra to the von Neumann algebra of a group  $G$ . More precisely we show the following.

**COROLLARY B.** *Let  $G$  be a torsion-free group which satisfies the Baum-Connes conjecture and let  $P$  be a finitely generated projective  $C_r^*G$ -module. Then the induced module  $NG \otimes_{C_r^*G} P$  is a free module, of rank equal the Kaplansky rank of  $P$*

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(which is an integer in this case), and the natural map

$$\tilde{K}_0(C_r^*G) \longrightarrow \tilde{K}_0(\mathcal{N}G)$$

is the zero-map.

Corollary B is reminiscent of some other change of ring statements. For instance, for any group  $G$  the map  $\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G)$  is predicted to be the zero map ([LR], Remark 3.17). Eckmann proved that for an arbitrary group  $G$ ,  $\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathcal{N}G)$  is the zero map (this follows from Proposition 3 of [E1] in conjunction with Remark A2 of [E2]); the same result was also proved by Schafer in [S] and in Lück's book [LU1] (Theorem 9.62). In [S] it is moreover shown that if  $\mathbb{C}G$  is contained in a division ring  $D \subset \mathcal{U}(G)$ , then  $\tilde{K}_0(\mathbb{C}G) \rightarrow \tilde{K}_0(\mathcal{N}G)$  is the zero map (here  $\mathcal{U}(G)$  denotes the Ore localization of  $\mathcal{N}G$  with respect to the set of non-zero divisors; equivalently  $\mathcal{U}(G)$  is the algebra of densely defined operators  $\ell^2(G) \rightarrow \ell^2(G)$ , affiliated to the von Neumann algebra  $\mathcal{N}G$  of  $G$ , see also Chapter 10 of [LU1]); for a survey on the relationship with Atiyah's conjecture on the rationality of  $\ell^2$ -Betti numbers of finite complexes, see [MV], pp. 63–64. Another related result is Theorem 6 of [E3], where Eckmann proves that for any torsion-free group  $G$ ,  $\tilde{K}_0(\mathbb{C}G) \rightarrow \tilde{K}_0(\mathcal{N}G)$  maps into the torsion subgroup. Finally, we like to mention that if the torsion-free group  $G$  of Corollary B is abelian,  $\mathcal{N}G = L^\infty(\hat{G})$  with  $\hat{G}$  the Pontryagin dual of  $G$  and as  $K_0(L^\infty(\hat{G})) = K_0(L(\hat{G}))$  with  $L(\hat{G})$  the algebra of measurable functions on  $\hat{G}$ , the result boils down to the well-known fact that continuous vector bundles over a connected compact space are measurably trivial.

Section 1 is devoted to a review of the algebras involved, as well as common examples of traces on them. We are mainly interested in the universal (Hattori-Stallings) trace on Banach algebras containing  $\mathbb{C}G$  as a subalgebra. In Section 2 we discuss the case of  $\ell^1G$  and Section 3 deals with group  $C^*$ -algebras. Section 4 is a further study of some elements in the image of some traces. As an application, we consider in Section 5 the  $C_{max}^*G$ -analogue of the following classical conjecture.

**CONJECTURE.** *Let  $G$  be a torsion-free group,  $P$  a finitely generated projective  $\mathbb{C}G$  module and  $\phi : G \rightarrow F$  a homomorphism, where  $F$  is a finite group. Then  $\mathbb{C}F \otimes_\phi P$  is a free  $\mathbb{C}F$ -module.*

**REMARK.** Corollary B can also be deduced from a version of the  $L^2$ -index theorem, which applies to the center-valued trace and which was proved by Lück in [LU2]. We thank the referee for pointing this out to us. Lück's arguments are analytic in nature, whereas our proof of Corollary B is purely algebraic, using suitable embeddings of groups into acyclic groups.

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## 1. Traces and Banach algebras

Our general setup for the sequel is as follows. Let  $G$  be a group and let  $\mathcal{A}G = \mathcal{A}$  denote a Banach algebra containing  $\mathbb{C}G$  as a subalgebra, with norm  $\|\cdot\|_{\mathcal{A}}$  therefore verifying  $\|f * g\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$  for all  $f, g \in \mathcal{A}$  (we write  $f * g$  for the product, to indicate that in case  $f, g \in \mathbb{C}G$ , this is just the convolution product). The first example is  $\ell^1G$ , the completion of  $\mathbb{C}G$  with respect to the  $\ell^1$ -norm, where we recall

that for an element  $a = \sum_g a_g g \in \mathbb{C}G$ , its  $\ell^1$ -norm is given by

$$\|a\|_1 = \sum_g |a_g|.$$

Similarly one defines  $\ell^2 G$ , the completion of  $\mathbb{C}G$  with respect to the  $\ell^2$ -norm, which for an element  $a = \sum_g a_g g \in \mathbb{C}G$ , is given by

$$\|a\|_2 = \sqrt{\sum_g |a_g|^2}.$$

This is very different from  $\ell^1 G$  because  $\ell^2 G$  is not an algebra anymore in most cases. Indeed, as soon as  $G$  is infinite the convolution product  $*$  on  $\mathbb{C}G$  doesn't extend to  $\ell^2 G$ . But  $\ell^2 G$  is a Hilbert space, with inner product given by

$$\langle a, b \rangle = \sum_{g \in G} \bar{a}_g b_g.$$

Via the right regular representation on  $\ell^2 G$ , the group algebra  $\mathbb{C}G$  may be viewed as a subalgebra of  $\mathcal{B}(\ell^2 G)$ , the  $C^*$ -algebra of bounded operators on  $\ell^2 G$ . The operator norm on  $\mathcal{B}(\ell^2 G)$  is then

$$\|a\|_{op} = \sup_{\|\xi\|_2=1} \|a(\xi)\|_2.$$

The closure of  $\mathbb{C}G$  for the operator norm as a subalgebra of  $\mathcal{B}(\ell^2 G)$  is called the *reduced  $C^*$ -algebra of  $G$*  and is denoted by  $C_r^* G$ . The von Neumann algebra  $\mathcal{N}G$  of  $G$  is the double commutant of  $\mathbb{C}G \subset \mathcal{B}(\ell^2 G)$ . It is a  $C^*$ -algebra, with norm the operator norm described above, that we denote by  $\|\cdot\|_{\mathcal{N}}$  when taken over  $\mathcal{N}G$  (we won't be considering any other topology on  $\mathcal{N}G$ ); note that the closure of  $\mathbb{C}G$  in  $\mathcal{N}G$  is  $C_r^* G$  as well. Any unitary representation  $\pi$  of  $G$  on a Hilbert space  $\mathcal{H}_\pi$  extends by linearity to a map  $\pi : \mathbb{C}G \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  into the bounded operators on  $\mathcal{H}_\pi$ , and the closure of  $\pi(\mathbb{C}G)$  for the operator norm is a  $C^*$ -algebra. The *maximal group  $C^*$ -algebra*  $C_{max}^* G$  is the completion of  $\mathbb{C}G$  with respect to the norm

$$\|a\|_{max} = \sup\{\|\pi(a)\|_{op}\},$$

the supremum being taken with respect to all unitary representations  $\pi$  of  $G$ . The norms we have discussed satisfy for  $x \in \mathbb{C}G$ ,

$$\|x\|_1 \geq \|x\|_{max} \geq \|x\|_{op} = \|x\|_{\mathcal{N}}$$

so that there are continuous algebra maps  $\ell^1 G \rightarrow C_{max}^* G \rightarrow C_r^* G \rightarrow \mathcal{N}G$ .

For  $\mathcal{B}$  any Banach algebra over  $\mathbb{C}$ , we write  $\overline{[\mathcal{B}, \mathcal{B}]}$  for the closure of the vector space  $[\mathcal{B}, \mathcal{B}] \subset \mathcal{B}$  generated by the elements of the form  $xy - yx$ ,  $x, y \in \mathcal{B}$ .

DEFINITION 1.1. Let  $\mathcal{B}$  be a Banach algebra over  $\mathbb{C}$ . We put

$$HH_0^{top}(\mathcal{B}) = \mathcal{B}/\overline{[\mathcal{B}, \mathcal{B}]}$$

for its *0-th topological Hochschild homology group*; we consider  $HH_0^{top}(\mathcal{B})$  as a topological  $\mathbb{C}$ -vector space with respect to the quotient Banach space structure.

Induced by the corresponding maps of Banach algebras, there are natural continuous maps

$$HH_0^{top}(\ell^1 G) \rightarrow HH_0^{top}(C_{max}^* G) \rightarrow HH_0^{top}(C_r^* G) \rightarrow HH_0^{top}(\mathcal{N}G)$$

which we will use later on. If  $V$  is a Hausdorff topological  $\mathbb{C}$ -vector space, a *continuous trace*  $\tau : \mathcal{B} \rightarrow V$  on the Banach algebra  $\mathcal{B}$  is a continuous  $\mathbb{C}$ -linear map satisfying  $\tau(xy) = \tau(yx)$ . The projection  $HS^{\mathcal{B}} : \mathcal{B} \rightarrow HH_0^{top}(\mathcal{B})$  is an example of such a trace; we call it the *Hattori-Stallings trace* and it is universal in the sense that every continuous trace  $\mathcal{B} \rightarrow V$  factors uniquely through it. For  $\tau : \mathcal{B} \rightarrow V$  a continuous trace, we will denote by same letter the induced map

$$\tau : K_0(\mathcal{B}) \rightarrow V.$$

For the convenience of the reader we recall its definition: if  $P$  is a finitely generated projective (left)  $\mathcal{B}$ -module, then  $P$  is isomorphic to a  $\mathcal{B}$ -module of the form  $\mathcal{B}^n \cdot A$  for some matrix  $A = (a_{ij}) \in M_n(\mathcal{B})$  satisfying  $A^2 = A$ . One then puts for  $[P] \in K_0(\mathcal{B})$

$$HS^{\mathcal{B}}([P]) = \sum a_{ii} + \overline{[\mathcal{B}, \mathcal{B}]} \in HH_0^{top}(\mathcal{B}),$$

and  $\tau([P]) = \sum \tau(a_{ii}) \in V$  is the image of  $HS^{\mathcal{B}}([P])$  under the natural map  $HH_0^{top}(\mathcal{B}) \rightarrow V$  induced by the continuous trace  $\tau$ .

Common examples of continuous traces include the *Kaplansky trace*  $\kappa : \mathcal{N}G \rightarrow \mathbb{C}$ , defined by  $\kappa(x) = \langle x, \delta_e, \delta_e \rangle$ , where  $\delta_e \in \ell^2 G$  is the element with coefficient 1 in  $e$  and 0 otherwise, yielding

$$\kappa : K_0(\mathcal{N}G) \longrightarrow \mathbb{C}.$$

We can define the Kaplansky trace  $\kappa : \mathcal{A}G \rightarrow \mathbb{C}$  for any Banach algebra completion  $\mathcal{A}G$  of  $\mathbb{C}G$  such that  $\| \cdot \|_{\mathcal{A}} \geq \| \cdot \|_{\mathcal{N}}$  on  $\mathbb{C}G$ . Indeed, in this case the identity map on  $\mathbb{C}G$  extend to a continuous map  $\mathcal{A}G \rightarrow \mathcal{N}G$ , and composing  $\mathcal{A}G \rightarrow \mathcal{N}G \rightarrow \mathbb{C}$  gives rise to  $\kappa : K_0(\mathcal{A}G) \rightarrow \mathbb{C}$ . For  $P$  a finitely generated projective  $\mathcal{A}G$  module,  $\kappa([P])$  is termed its *Kaplansky rank*. An enhanced version of the Kaplansky trace is the *center-valued trace*, which we describe now. Let  $Z(\mathcal{N}G)$  denote the center of  $\mathcal{N}G$ ; it is a commutative  $C^*$ -subalgebra. The *center-valued trace* is the unique continuous trace  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  which restricts to the identity on  $Z(\mathcal{N}G)$ ; it satisfies  $\|ctr(x)\|_{\mathcal{N}} \leq \|x\|_{\mathcal{N}}$  and is universal in the sense that every continuous trace  $\mathcal{N}G \rightarrow V$  into a Hausdorff topological  $\mathbb{C}$ -vector space  $V$  factors uniquely through the center-valued trace (for the definition and basic properties of the center-valued trace on a finite von Neumann algebra, see [KR]). In particular, the Kaplansky trace factors as  $\mathcal{N}G \rightarrow Z(\mathcal{N}G) \rightarrow \mathbb{C}$ , and  $ctr = \kappa$  when  $Z(\mathcal{N}G) = \mathbb{C}$  (that is, when  $\mathcal{N}G$  is a *factor*). Again, if  $\mathcal{A}G$  denotes a Banach algebra completion of  $\mathbb{C}G$  with  $\| \cdot \|_{\mathcal{A}} \geq \| \cdot \|_{\mathcal{N}}$ , there is a continuous map  $\mathcal{A}G \rightarrow \mathcal{N}G$  and  $ctr$  gives rise to

$$ctr : K_0(\mathcal{A}G) \rightarrow Z(\mathcal{N}G);$$

this map factors by naturality as  $K_0(\mathcal{A}G) \rightarrow HH_0^{top}(\mathcal{A}G) \rightarrow HH_0^{top}(\mathcal{N}G) \rightarrow Z(\mathcal{N}G)$ . Because of the universal property of  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  one has a natural isomorphism  $HH_0^{top}(\mathcal{N}G) \cong Z(\mathcal{N}G)$  and can write:

$$ctr : K_0(\mathcal{A}G) \longrightarrow HH_0^{top}(\mathcal{N}G) = Z(\mathcal{N}G).$$

More precisely the following holds.

LEMMA 1.2. *For any group  $G$ ,  $[\mathcal{N}G, \mathcal{N}G] = \overline{[\mathcal{N}G, \mathcal{N}G]} \subset \mathcal{N}G$  and one has a natural decomposition as Banach spaces*

$$\mathcal{N}G = Z(\mathcal{N}G) \oplus [\mathcal{N}G, \mathcal{N}G].$$

The center-valued trace  $\mathcal{N}G \rightarrow Z(\mathcal{N}G)$  induces an isomorphism of Banach spaces

$$HH_0^{top}(\mathcal{N}G) \rightarrow Z(\mathcal{N}G),$$

with inverse given by the restriction of the canonical map  $\mathcal{N}G \rightarrow HH_0^{top}(\mathcal{N}G)$  to  $Z(\mathcal{N}G)$ . Under these natural isomorphisms  $HH_0^{top}(\mathcal{N}G) \cong Z(\mathcal{N}G)$ , the element  $HS^{\mathcal{N}G}([\mathcal{N}G])$  corresponds to  $1_{\mathcal{N}G} \in Z(\mathcal{N}G)$ .

PROOF. Let  $K$  and  $I$  stand for the kernel and the image of  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  respectively. Since  $x = ctr(x) + (x - ctr(x))$  for  $x \in \mathcal{N}G$ , we see that  $I + K = \mathcal{N}G$ . Now let  $y \in I \cap K$ . Then  $0 = ctr(y) = y$ , thus  $\mathcal{N}G = I \oplus K$ . Note that the subspaces  $I = \ker(id - ctr)$  and  $K = \ker(ctr)$  are closed in  $\mathcal{N}G$ , each being the kernel of a continuous map. Obviously, because  $ctr = id$  on  $Z(\mathcal{N}G)$ ,  $I = Z(\mathcal{N}G)$  and it remains to show that  $K = [\mathcal{N}G, \mathcal{N}G]$ . Since  $ctr$  is a continuous trace,  $[\mathcal{N}G, \mathcal{N}G] \subset K$ ; conversely, if  $x \in K = \ker(ctr)$  then by a result of Fack and de la Harpe ([FH], Théorème 3.2)  $x$  is the sum of 10 commutators in  $\mathcal{N}G$  so that  $x \in [\mathcal{N}G, \mathcal{N}G]$ . It follows that  $[\mathcal{N}G, \mathcal{N}G] = \overline{[\mathcal{N}G, \mathcal{N}G]} = K$ , and thus  $Z(\mathcal{N}G)$  is naturally isomorphic to  $HH_0^{top}(\mathcal{N}G)$ . For the last statement of the lemma, we observe that  $[\mathcal{N}G] \in K_0(\mathcal{N}G)$  is represented by the idempotent  $1_{\mathcal{N}G}$ ; thus by the definition of the Hattori-Stallings trace one has  $HS^{\mathcal{N}G}([\mathcal{N}G]) = 1_{\mathcal{N}G} + \overline{[\mathcal{N}G, \mathcal{N}G]}$ , and because  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  maps  $1_{\mathcal{N}G}$  to  $1_{Z(\mathcal{N}G)} = 1_{\mathcal{N}G}$  the claim follows.  $\square$

The augmentation trace  $\epsilon : C_{max}^*G \rightarrow \mathbb{C}$  is the  $C_{max}^*$ -algebra map induced by  $G \rightarrow \{e\}$ . On the dense subalgebra  $\mathbb{C}G$ , this is just the ordinary augmentation  $\epsilon(a) = \sum_g a_g$  for  $a = \sum_g a_g g$ , yielding

$$\epsilon : K_0(C_{max}^*G) \longrightarrow \mathbb{C}.$$

More generally, we define the augmentation trace  $\epsilon : \mathcal{A}G \rightarrow \mathbb{C}$  for any Banach algebra completion  $\mathcal{A}G$  of  $\mathbb{C}G$  as long as  $\|\cdot\|_{\mathcal{A}} \geq \|\cdot\|_{max}$  on  $\mathbb{C}G$ , by composing in the obvious way.

To put Theorem A of the Introduction into a more general framework, the following definition is convenient.

DEFINITION 1.3. Let  $\mathcal{A}G$  be a Banach algebra containing  $\mathbb{C}G$  as a subalgebra and denote by  $\mathbb{C}G_f$  the  $\mathbb{C}$ -vector space with basis the elements of finite order in  $G$ .

(i) A subset  $S \subset HH_0^{top}(\mathcal{A}G)$  is called *f-supported*, if it lies in the image of  $\mathbb{C}G_f \subset \mathcal{A}G$  under the natural quotient map  $\mathcal{A}G \rightarrow HH_0^{top}(\mathcal{A}G)$ .

(ii) The Banach algebra  $\mathcal{A}G$  has the *f-Trace Property* if the image of the Hattori-Stallings trace

$$HS^{\mathcal{A}} : K_0(\mathcal{A}G) \rightarrow HH_0^{top}(\mathcal{A}G)$$

is f-supported.

(iii) A finitely generated projective  $\mathcal{A}G$ -module  $P$  is called *f-supported* if the element  $HS^{\mathcal{A}}(P) \in HH_0^{top}(\mathcal{A}G)$  is.

For the reduced  $C^*$ -algebra  $C_r^*G$  to have the f-Trace Property can be viewed as an analogue of the Bass conjecture for  $\mathbb{C}G$ , which asserts that the Hattori-Stallings trace  $K_0(\mathbb{C}G) \rightarrow HH_0(\mathbb{C}G)$  maps into the subspace spanned by the images of the elements of finite order in  $G$  (cf. [B]). Theorem A of the Introduction has the following equivalent formulation.

**THEOREM 1.4.** *Suppose that  $G$  satisfies the Baum-Connes conjecture. Then  $C_r^*G$  has the f-Trace Property.*

The proof will be given in Section 3. Theorem 1.4 gives evidences that  $C_r^*G$  should have the f-Trace Property for any group  $G$ . The situation is quite different for the maximal  $C^*$ -algebra: in Section 3 we will show that for infinite groups with Kazhdan's property (T), the algebra  $C_{max}^*G$  does not have the f-Trace Property.

## 2. The $\ell^1$ case and the Bass conjecture

In case of the Banach algebra  $\ell^1G$ , one has the following simple description of  $HH_0^{top}(\ell^1G)$ . We write  $\ell^1[G]$  for the Banach space of  $\ell^1$ -function  $[G] \rightarrow \mathbb{C}$ , where  $[G]$  stands for the set of conjugacy classes of  $G$ ; the Banach space structure is given by the  $\ell^1$ -norm. There is a natural projection  $\theta : \ell^1G \rightarrow \ell^1[G]$  obtained by summing over conjugacy classes, which is a continuous trace (see below), so that  $\theta$  factors through  $HH_0^{top}(\ell^1G)$ .

**LEMMA 2.1.** *For any group  $G$ , the natural map  $\theta : \ell^1G \rightarrow \ell^1[G]$  induces an isomorphism of Banach spaces*

$$\theta_* : HH_0^{top}(\ell^1G) \rightarrow \ell^1[G].$$

**PROOF.** The projection  $\theta : \ell^1G \rightarrow \ell^1[G]$  is continuous, because it is norm decreasing. Let us first show that  $[\ell^1G, \ell^1G]$  is contained in  $\ker(\theta)$ . For any element  $f = \sum f_{gg} \in \ell^1G$ , we write

$$\theta_{[g]}f := \sum_{x \in [g]} f_x \in \mathbb{C}$$

for the  $[g]$ -coefficient of  $\theta(f)$ . The elements of  $[\ell^1G, \ell^1G]$  are finite linear combinations of commutators  $ab - ba$ , with  $a, b \in \ell^1G$ . So, for an element  $ab - ba$ , if we write  $a = \sum a_x x$ ,  $b = \sum b_y y$  and  $ab = \sum c_{gg} \in \ell^1G$ , then we have that  $c_g = \sum_{xy=g} a_x b_y$ , so that

$$\theta_{[g]}(ab) = \sum_{z \in [g]} \sum_{xy=z} a_x b_y$$

which is equal to  $\theta_{[g]}(ba)$ , because  $ba = \sum_{h \in G} (\sum_{uv=h} b_u a_v) h$  and  $\sum_{uv=h} b_u a_v = \sum_{vu=u^{-1}hu} a_v b_u$ . It follows that  $[\ell^1G, \ell^1G]$  lies in  $\ker(\theta)$  and therefore its closure as well since  $\theta$  is continuous. Thus  $\theta$  induces a continuous surjection  $\theta_* : HH_0^{top}(\ell^1G) \rightarrow \ell^1[G]$ .

To see that  $\theta_*$  is injective, take  $a \in \ell^1G$  and assume that  $\theta(a) = 0$ . We write  $a = \sum a_{[g]}$ , where  $a_{[g]} \in \ell^1G$  is supported on the conjugacy class  $[g] \subset G$ . Since  $\theta_{[g]}(a) = \theta_{[g]}(a_{[g]})$ , we see that  $\theta_{[g]}(a_{[g]}) = 0$  as well. Now  $a_{[g]}$  can be written as  $\lim s_n$  with  $s_n \in \mathbb{C}G$ , and  $s_n$  having its support in  $[g]$ . Let  $e_n := \theta_{[g]}(s_n) \in \mathbb{C}$ . Clearly  $\lim e_n = 0 \in \mathbb{C}$ . We now choose an element  $g_0 \in [g]$  and write

$$r_n := s_n - e_n g_0,$$

which is an element in  $\mathbb{C}G$  with its support in  $[g]$ ; because its augmentation is zero and it is supported on a single conjugacy class, the element maps to 0 in  $HH_0(\mathbb{C}G)$  and is therefore a finite sum of commutators in  $\mathbb{C}G$ :

$$r_n = \sum_i q_{n,i} w_{n,i} - w_{n,i} q_{n,i}$$

and, in  $\ell^1 G$ ,  $\lim r_n = \lim s_n - \lim e_n g_0 = a_{[g]} - 0$  so that  $a_{[g]}$  lies in the closure of  $[\mathbb{C}G, \mathbb{C}G]$  in  $\ell^1 G$ . It then follows that  $\sum_{[g]} a_{[g]} = a$  lies in the closure of  $[\mathbb{C}G, \mathbb{C}G]$  as well.

The inverse of  $\theta_*$  is given as follows. One picks for every  $[g]$  a representative  $\tilde{g} \in G$  and puts

$$\theta_*^{-1}\left(\sum c_{[g]}[g]\right) := \sum c_{[g]}\tilde{g} + \overline{[\ell^1 G, \ell^1 G]},$$

which defines a norm preserving inverse to  $\theta_*$ , concluding the proof.  $\square$

REMARK 2.2. The argument shows that in  $\ell^1 G$ , the closure of  $[\mathbb{C}G, \mathbb{C}G]$  equals the closure of  $[\ell^1 G, \ell^1 G]$ ; we do not know whether  $[\ell^1 G, \ell^1 G]$  is already closed in  $\ell^1 G$ .

In [BCM] we worked with the trace  $\theta : \ell^1 G \rightarrow \ell^1[G]$  and we factored it through the algebraic  $HH_0(\ell^1 G) = \ell^1 G / [\ell^1 G, \ell^1 G]$ . In view of the previous lemma, the “ $\ell^1$  Bass Conjecture for  $G$ ” in the sense of Conjecture 2.2 of [BCM] is precisely the “ $f$ -Trace Property for  $\ell^1 G$ ” in the sense of our Definition 1.3. The main result of [BCM] can therefore be reformulated as follows.

THEOREM 2.3. *For an arbitrary group  $G$ , the image of the composite map*

$$K_0^G(\underline{EG}) \rightarrow K_0(\ell^1 G) \rightarrow HH_0^{\text{top}}(\ell^1 G)$$

*is  $f$ -supported. If  $G$  satisfies the Bost conjecture then the image of the Hattori-Stallings trace*

$$HS^{\ell^1} : K_0(\ell^1 G) \rightarrow HH_0^{\text{top}}(\ell^1 G) = \ell^1[G]$$

*is  $f$ -supported, i.e.  $\ell^1 G$  has the  $f$ -Trace Property.*

### 3. Some $C^*$ -algebra cases

To prove Theorem A of the introduction we actually prove a more general result. We write  $\mu_{\ell^1} : K_0^G(\underline{EG}) \rightarrow K_0(\ell^1 G)$  for the Bost assembly map (see Lafforgue [L] for its definition). More generally, for  $\mathcal{A}G$  a Banach algebra completion of  $\mathbb{C}G$  satisfying  $\| \cdot \|_1 \geq \| \cdot \|_{\mathcal{A}}$ , we write  $\mu_{\mathcal{A}} : K_0^G(\underline{EG}) \rightarrow K_0(\mathcal{A}G)$  for the composite map  $i_* \circ \mu_{\ell^1}$ , where  $i_* : K_0(\ell^1 G) \rightarrow K_0(\mathcal{A}G)$  is induced by the natural map  $i : \ell^1 G \rightarrow \mathcal{A}G$ . It is proved in [L] that in this notation,  $\mu_{C_r^*}$  agrees with the Baum-Connes assembly map.

PROPOSITION 3.1. *Let  $G$  be a countable group and  $\mathcal{A}G$  a Banach algebra completion of  $\mathbb{C}G$  such that  $\|x\|_1 \geq \|x\|_{\mathcal{A}}$  for all  $x \in \mathbb{C}G$ . If the assembly map*

$$\mu_{\mathcal{A}} : K_0^G(\underline{EG}) \rightarrow K_0(\mathcal{A}G)$$

*is surjective, then  $\mathcal{A}G$  has the  $f$ -Trace Property.*

PROOF. The assumption on  $\| \cdot \|_{\mathcal{A}}$  implies that there is a continuous map of completions  $\ell^1 G \rightarrow \mathcal{A}G$ . By Theorem 2.3 the image of

$$K_0^G(\underline{EG}) \rightarrow K_0(\ell^1 G) \rightarrow HH_0^{\text{top}}(\ell^1 G)$$

is  $f$ -supported. Our claim then follows by a diagram chase in the following commutative diagram

$$\begin{array}{ccccc}
 & & K_0(\ell^1 G) & \xrightarrow{HS^{\ell^1}} & HH_0^{\text{top}}(\ell^1 G) \\
 & \nearrow^{\mu_{\ell^1}} & \downarrow & & \downarrow \\
 K_0^G(\underline{EG}) & & & & \mathbb{C}G_f \\
 & \searrow_{\mu_{\mathcal{A}}} & & & \swarrow \\
 & & K_0(\mathcal{A}G) & \xrightarrow{HS^{\mathcal{A}}} & HH_0^{\text{top}}(\mathcal{A}G).
 \end{array}$$

□

PROOF OF THEOREM A AND THEOREM 1.4. Since the operator norm on  $\mathbb{C}G$  is always bounded by the  $\ell^1$ -norm, there is a natural continuous map  $\ell^1 G \rightarrow C_r^* G$ . The assumption in Theorem A (resp. Theorem 1.4) guarantee that the assembly map in question is surjective and we can apply Proposition 3.1 with  $\mathcal{A}G = C_r^* G$  to conclude the proof. □

All we actually do concerns elements lying in the image of the assembly map. For elements which do not lie in the image of the assembly map, the situation can be very different, as we show now.

A group  $G$  is called  $a$ -( $T$ )-menable if it admits a metrically proper affine action on a Hilbert space, see [CCJJV] for an introduction. A group  $G$  has Kazhdan's property ( $T$ ) if any affine action on a Hilbert space has a globally fixed point, see [HV] for an introduction. For a list of groups in these classes see [MV], [CCJJV] and [V2].

PROPOSITION 3.2. (a) Let  $G$  be a torsion-free infinite, countable group with property ( $T$ ). Then  $C_{\max}^* G$  does not have the  $f$ -Trace Property.

(b) Let  $G$  be an  $a$ -( $T$ )-menable group. Then  $C_{\max}^* G$  has the  $f$ -Trace Property.

PROOF. (a) Let  $G$  be a torsion-free group with Kazhdan's property ( $T$ ). Then there exist an idempotent  $\pi \in C_{\max}^* G$  (the Kazhdan projection) which, on a unitary representation of  $G$  acts via the projection onto the  $G$ -invariant subspace; for a construction of  $\pi$ , see [V1] or 3.9.17 of [HR]. Thus it acts via  $id$  on the the trivial representation and as the zero operator on  $\ell^2 G$  (we use here the fact that  $G$  is an infinite group and therefore the  $G$ -invariant subspace of  $\ell^2 G$  equals  $\{0\}$ ). It follows that  $\epsilon(\pi) = 1 \in \mathbb{C}$  and  $\kappa(\pi) = 0 \in \mathbb{C}$ . Because for the unit element  $e \in C_{\max}^* G$  one has  $\kappa(e) = \epsilon(e) = 1$  and by considering the projective modules  $[C_{\max}^* G \cdot \pi]$  resp.  $[C_{\max}^* G]$  corresponding to the idempotents  $\pi, e \in C_{\max}^* G$ , we see that the  $\mathbb{C}$ -vector space spanned by the image of the Hattori-Stallings trace  $HS^{C_{\max}^*} : K_0(C_{\max}^* G) \rightarrow HH_0^{\text{top}}(C_{\max}^* G)$  must be at least 2-dimensional. Therefore,  $C_{\max}^* G$  does not have the  $f$ -Trace Property.

(b) According to [HK], the assumptions imply that the assembly map  $\mu_{C_{\max}^* G}$  is surjective so that the hypothesis of Proposition 3.1 is satisfied for  $\mathcal{A}G = C_{\max}^* G$ , yielding the conclusion of (b). □



#### 4. Properties of $f$ -supported elements

Let  $\mathcal{A}G$  be a Banach algebra completion of  $\mathbb{C}G$  such that  $\|\cdot\|_{\mathcal{A}} \geq \|\cdot\|_{\mathcal{N}}$ . Then the Kaplansky trace is defined on  $\mathcal{A}G$ , as explained earlier. Let  $\iota_G : \mathbb{C} \rightarrow HH_0^{\text{top}}(\mathcal{A}G)$  be the injective map  $\lambda \mapsto \lambda \cdot HS^{\mathcal{A}}([\mathcal{A}G])$ . We write  $\tilde{\kappa}$  for the continuous trace given by the composite map

$$\tilde{\kappa} : \mathcal{A}G \xrightarrow{\kappa} \mathbb{C} \xrightarrow{\iota_G} HH_0^{\text{top}}(\mathcal{A}G),$$

and, following our convention, we use the same notation for the induced maps on  $K$ -theory

$$\tilde{\kappa} : K_0(\mathcal{A}G) \rightarrow HH_0^{\text{top}}(\mathcal{A}G), \quad [P] \mapsto \kappa([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G]).$$

In the torsion-free case, the  $f$ -Trace Property for  $\mathcal{A}G$  implies that  $HS^{\mathcal{A}} : K_0(\mathcal{A}G) \rightarrow HH_0^{\text{top}}(\mathcal{A}G)$  maps into the one-dimensional subspace  $\iota_G(\mathbb{C})$  spanned by  $HS^{\mathcal{A}}([\mathcal{A}G])$ . This is used in the following.

**PROPOSITION 4.1.** *Let  $G$  be a torsion-free group and  $\mathcal{A}G$  a Banach algebra completion of  $\mathbb{C}G$  satisfying  $\|\cdot\|_{\mathcal{A}} \geq \|\cdot\|_{\mathcal{N}}$ . For  $P$  a finitely generated projective and  $f$ -supported  $\mathcal{A}G$ -module,*

$$HS^{\mathcal{A}}([P]) = \kappa([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G]) = \tilde{\kappa}([P]).$$

*In particular, if  $\mathcal{A}G$  has the  $f$ -Trace Property, then*

$$HS^{\mathcal{A}} = \tilde{\kappa} : K_0(\mathcal{A}G) \rightarrow HH_0^{\text{top}}(\mathcal{A}G).$$

**PROOF.** Because  $\|\cdot\|_{\mathcal{A}} \geq \|\cdot\|_{\mathcal{N}}$ , the Kaplansky trace  $\kappa : \mathcal{A}G \rightarrow \mathbb{C}$  is defined and it factors through  $HH_0^{\text{top}}(\mathcal{A}G)$ , yielding a commutative diagram

$$\begin{array}{ccc} \mathcal{A}G & \xrightarrow{\kappa} & \mathbb{C} \\ & \searrow HS^{\mathcal{A}} & \nearrow \bar{\kappa} \\ & & HH_0^{\text{top}}(\mathcal{A}G). \end{array}$$

Let  $P$  be a finitely generated projective  $\mathcal{A}G$ -module. Since  $G$  is torsion-free and  $P$  is  $f$ -supported one has  $HS^{\mathcal{A}}([P]) = \lambda([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G])$  for some  $\lambda([P]) \in \mathbb{C}$ . By applying  $\bar{\kappa}$ , we find

$$\bar{\kappa}(HS^{\mathcal{A}}([P])) = \kappa([P])$$

and, because  $HS^{\mathcal{A}}([P]) = \lambda([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G])$ ,

$$\bar{\kappa}(HS^{\mathcal{A}}([P])) = \bar{\kappa}(\lambda([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G])) = \lambda([P]).$$

It follows that  $\lambda([P]) = \kappa([P])$  and therefore  $HS^{\mathcal{A}}([P]) = \kappa([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G]) = \tilde{\kappa}([P])$  as claimed.  $\square$

The following is reminiscent of the weak Bass conjecture for  $\mathbb{C}G$ , and implies it in certain cases.

COROLLARY 4.2. *Let  $G$  be a torsion-free group and  $\mathcal{A}G$  a Banach algebra completion of  $\mathbb{C}G$  satisfying  $\| \cdot \|_1 \geq \| \cdot \|_{\mathcal{A}} \geq \| \cdot \|_{max}$ . If  $P$  denotes a finitely generated projective  $\mathcal{A}G$ -module that is in the image of the assembly map  $\mu_{\mathcal{A}}$  then  $\epsilon([P]) = \kappa([P])$ . In particular, if  $\mu_{\mathcal{A}}$  is surjective, then*

$$\epsilon = \kappa : K_0(\mathcal{A}G) \rightarrow \mathbb{C}.$$

PROOF. Since  $\| \cdot \|_{\mathcal{A}} \geq \| \cdot \|_{max}$  the augmentation trace  $\epsilon : \mathcal{A}G \rightarrow \mathbb{C}$  is defined and factors through  $HH_0^{top}(\mathcal{A}G)$ , yielding a commutative diagram

$$\begin{array}{ccc} \mathcal{A}G & \xrightarrow{\epsilon} & \mathbb{C} \\ & \searrow HS^{\mathcal{A}} & \nearrow \bar{\epsilon} \\ & & HH_0^{top}(\mathcal{A}G) \end{array}$$

so that

$$\bar{\epsilon}(HS^{\mathcal{A}}([P])) = \epsilon([P]).$$

Because  $\| \cdot \|_1 \geq \| \cdot \|_{\mathcal{A}}$  the assembly map  $\mu_{\mathcal{A}}$  is defined and we know from Proposition 3.1 that the surjectivity of  $\mu_{\mathcal{A}}$  implies that  $\mathcal{A}G$  has the f-Trace Property. Hence, as  $\| \cdot \|_{\mathcal{A}} \geq \| \cdot \|_{max} \geq \| \cdot \|_{\mathcal{N}}$ , we can apply the previous proposition to conclude that

$$HS^{\mathcal{A}}([P]) = \kappa([P]) \cdot HS^{\mathcal{A}}([\mathcal{A}G])$$

so that

$$\epsilon([P]) = \bar{\epsilon}(HS^{\mathcal{A}}([P])) = \kappa([P]) \cdot \bar{\epsilon}(HS^{\mathcal{A}}([\mathcal{A}G])) = \kappa([P]),$$

yielding the result claimed.  $\square$

The following is a particular case of Corollary 4.2 for the maximal  $C^*$ -algebra  $C_{max}^*G$ , which will be used later on.

COROLLARY 4.3. *Let  $G$  be a torsion-free group. Then for any finitely generated projective  $C_{max}^*G$ -module  $P$  with  $[P] \in K_0(C_{max}^*G)$  in the image of the assembly map,*

$$HS^{C_{max}^*}([P]) = \kappa([P]) \cdot HS^{C_{max}^*}(C_{max}^*G) \in HH_0^{top}(C_{max}^*G)$$

and

$$\kappa([P]) = \epsilon([P]).$$

Another application concerns the center-valued trace.

PROPOSITION 4.4. *Let  $G$  be a torsion-free group and assume that  $C_r^*G$  has the f-Trace Property. Then the center-valued trace*

$$ctr : K_0(C_r^*G) \rightarrow Z(\mathcal{N}G)$$

satisfies  $ctr([P]) = \kappa([P]) \cdot 1_{\mathcal{N}G}$  for any finitely generated projective  $C_r^*G$ -module  $P$ , where  $1_{\mathcal{N}G} \in \mathcal{N}G$  is the unit element.

PROOF. As explained in Section 1, the center-valued trace is defined on  $K_0(C_r^*G)$  as the composite map

$$ctr : K_0(C_r^*G) \rightarrow K_0(\mathcal{N}G) \rightarrow HH_0^{top}(\mathcal{N}G) = Z(\mathcal{N}G)$$

and, by naturality, can be factored as

$$K_0(C_r^*G) \rightarrow HH_0^{top}(C_r^*G) \xrightarrow{i_*} HH_0^{top}(\mathcal{N}G) = Z(\mathcal{N}G),$$

with  $i_*$  induced by the inclusion  $C_r^*G \subset \mathcal{N}G$ . It follows from Proposition 4.1 that  $ctr([P]) = i_*(\kappa[P] \cdot HS^{C_r^*G}([C_r^*G]))$ , and under our identification of  $HH_0^{top}(\mathcal{N}G)$  with  $Z(\mathcal{N}G)$ ,  $i_*(HS^{C_r^*G}([C_r^*G])) = HS^{\mathcal{N}G}([\mathcal{N}G])$  corresponds by Lemma 1.2 to  $1_{\mathcal{N}G} \in Z(\mathcal{N}G)$ . It follows that for  $[P] \in K_0(C_r^*G)$ ,  $ctr([P]) = \kappa([P]) \cdot 1_{\mathcal{N}G}$ .  $\square$

We now prove Corollary B of the Introduction. Recall that for a ring  $R$  with 1 the reduced group  $\tilde{K}_0(R)$  is defined as the cokernel of the natural map  $K_0(\mathbb{Z}) \rightarrow K_0(R)$ .

PROOF OF COROLLARY B. It is a basic fact that for any group  $G$ , the center-valued trace  $ctr : K_0(\mathcal{N}G) \rightarrow Z(\mathcal{N}G)$  is injective (this follows from Theorems 8.4.3 and 8.4.4 of [KR]). Since  $G$  is assumed to satisfy the Baum-Connes conjecture and is torsion-free, the finitely generated projective  $C_r^*G$ -module  $P$  has  $\kappa(P) \in \mathbb{N}$  (for a direct proof of this fact, not using Atiyah's  $L^2$ -Index Theorem, see [MV]). By Proposition 4.4 we infer that  $ctr([P]) = ctr([C_r^*G^n])$ , where  $n = \kappa([P])$ . Passing to  $K_0(\mathcal{N}G)$  we conclude that  $\mathcal{N}G \otimes_{C_r^*G} P$  is stably isomorphic to  $\mathcal{N}G^n$  and it follows that the change of ring map  $\tilde{K}_0(C_r^*G) \rightarrow \tilde{K}_0(\mathcal{N}G)$  is trivial. Because two stably isomorphic finitely generated projective  $\mathcal{N}G$ -modules are isomorphic (cf. [KR] loc. cit.), it follows that for  $P$  finitely generated projective over  $C_r^*G$ , the module  $\mathcal{N}G \otimes_{C_r^*G} P$  is not only stably free, but free of rank  $\kappa([P])$ .  $\square$

The following variation of Corollary B is easier to prove and applies also to not necessarily torsion-free groups. We recall that  $\mathcal{N}G$  is a *factor* if and only if  $Z(\mathcal{N}G) = \mathbb{C}$ , which is known to be equivalent with saying that for every  $g \in G \setminus \{e\}$  the centralizer of  $g$  in  $G$  has infinite index; such groups are also known as ICC-groups (for a survey on group von Neumann algebras see Lück's book [LU1]).

COROLLARY 4.5. *Let  $G$  be a group satisfying the Baum-Connes conjecture and assume that  $\mathcal{N}G$  is a factor. Then*

$$\tilde{K}_0(C_r^*G) \longrightarrow \tilde{K}_0(\mathcal{N}G)$$

*maps into the torsion subgroup.*

PROOF. Since  $\mathcal{N}G$  is a factor,  $ctr = \kappa$  (see Section 1). By assumption, the group  $G$  satisfies the Baum-Connes conjecture, and hence according to [LU2] the Kaplansky trace  $\kappa : K_0(C_r^*G) \rightarrow \mathbb{C}$  maps into  $\mathbb{Q}$  (see also [BCM] for an alternate proof). It follows that if  $P$  is a finitely generated projective  $C_r^*G$ -module with  $\kappa([P]) = m/n$  for some  $m, n \in \mathbb{N}$ ,  $(\mathcal{N}G \otimes_{C_r^*G} P)^n$  is isomorphic to  $\mathcal{N}G^m$ , showing that the image  $[\mathcal{N}G \otimes_{C_r^*G} P]$  in  $\tilde{K}_0(\mathcal{N}G)$  is a torsion element.  $\square$

We want to give an example to illustrates that the Hattori-Stallings trace

$$HS^{C_r^*} : K_0(C_r^*G) \rightarrow HH_0^{top}(C_r^*G)$$

detects in general more K-theory classes than the center-valued trace

$$ctr : K_0(C_r^*G) \rightarrow Z(\mathcal{N}G).$$

To this end, we recall that the center-valued trace  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  has the following two basic properties: first, for  $x \in Z(\mathcal{N}G)$  and  $y \in \mathcal{N}G$  one has  $ctr(xy) = x \cdot ctr(y)$ , and second, because of the universal property of  $ctr$ ,  $\kappa(ctr(x)) = \kappa(x)$  for any  $x \in \mathcal{N}G$ . As a consequence, the following Lemma holds (see also Emmanouil [EM] for a different proof).

LEMMA 4.6. *Let  $g \in G$  with  $[G : C_G(g)] = \infty$ . Then the center-valued trace  $ctr : \mathcal{N}G \rightarrow Z(\mathcal{N}G)$  satisfies  $ctr(g) = 0$ .*

PROOF. Using the usual embedding  $\mathcal{N}G \rightarrow \ell^2 G$ , we can write  $ctr(g) = \sum c_u u$ . Since  $\kappa(u^{-1}ctr(g)) = c_u$ , we need to show that  $\kappa(h \cdot ctr(g)) = 0$  for all  $h \in G$ . This is certainly so if the cardinality of the conjugacy class of  $h$ ,  $[G : C_G(h)]$  is infinite, because  $v \cdot ctr(g) = ctr(g) \cdot v$  for all  $v \in G$  so that  $c_u = c_{v^{-1}uv}$  and thus  $c_u = 0$  if  $u$  has infinitely many distinct conjugates. In case  $[G : C_G(h)] < \infty$ , we have with  $h_\mu := \frac{1}{[G : C_G(h)]} \sum_{t \in [h]} t \in Z(\mathcal{N}G)$ :

$$\kappa(h \cdot ctr(g)) = \kappa(h_\mu \cdot ctr(g)) = \kappa(ctr(h_\mu g)) = \kappa(h_\mu g) = 0$$

because for  $t \in [h]$ ,  $tg \neq e$ , otherwise  $g$  would have only finitely many conjugates.  $\square$

EXAMPLE 4.7. *Let  $D_\infty$  denote the infinite dihedral group. Then one has  $K_0(C_r^*D_\infty) \cong \mathbb{Z}^3$  and the image of  $ctr : K_0(C_r^*D_\infty) \rightarrow Z(\mathcal{N}G)$  is isomorphic to  $\mathbb{Z}$ , generated by  $\frac{1}{2} \cdot 1_{\mathcal{N}G}$ . On the other hand,  $HS^{C_r^*} : K_0(C_r^*D_\infty) \rightarrow HH_0^{top}(C_r^*D_\infty)$  is injective.*

PROOF. Let  $D_\infty = \langle x \rangle * \langle y \rangle$ , generated by the two involutions  $x, y$ . Then by using the formula for the K-theory of the reduced group C\*-algebra of a free product of groups (a special case of the Pimsner sequence for groups acting on trees [P]), and using that for a finite groups  $G$  one has  $C_r^*(G) = \mathbb{C}G$  and  $K_0(\mathbb{C}G) = R_{\mathbb{C}}(G)$ , the complex representation ring, we have a surjective map  $R_{\mathbb{C}}(\langle x \rangle) \oplus R_{\mathbb{C}}(\langle y \rangle) \rightarrow K_0(C_r^*D_\infty)$  inducing  $\tilde{K}_0(\mathbb{C}\langle x \rangle) \oplus \tilde{K}_0(\mathbb{C}\langle y \rangle) \cong \tilde{K}_0(C_r^*D_\infty) \cong \mathbb{Z}^2$ . A  $\mathbb{Z}$ -basis for  $K_0(C_r^*D_\infty)$  is then given by  $[C_r^*D_\infty]$ ,  $[P]$  and  $[Q]$ , where P and Q are the projective modules induced up from the trivial  $\mathbb{C}\langle x \rangle$ -module  $\mathbb{C}$ , resp. the trivial  $\mathbb{C}\langle y \rangle$ -module  $\mathbb{C}$ . Because  $P = C_r^*D_\infty \cdot e_P$  with  $e_P$  the idempotent  $\frac{1}{2}(1+x)$  in  $C_r^*D_\infty$  and because the centralizer of  $x \in D_\infty$  has infinite index, we conclude from Lemma 4.6 that  $ctr([P]) = (\frac{1}{2} + 0) \cdot 1_{\mathcal{N}} \in Z(\mathcal{N}D_\infty)$  and similarly for  $[Q]$ . As a result,  $ctr([P] - [Q]) = 0$  so that  $ctr : K_0(C_r^*D_\infty) \rightarrow Z(\mathcal{N}D_\infty)$  is not injective and the image is a free abelian group generated by  $\frac{1}{2} \cdot 1_{\mathcal{N}G}$  as claimed. On the other hand, the projection  $p : D_\infty \rightarrow \langle x \rangle \oplus \langle y \rangle$  yields a trace  $t_p : C_r^*D_\infty \rightarrow \mathbb{C}(\langle x \rangle \oplus \langle y \rangle)$ , resp.  $t_p : K_0(C_r^*D_\infty) \rightarrow \mathbb{C}(\langle x \rangle \oplus \langle y \rangle)$ , satisfying

$$t_p([P]) = (1+x)/2, \quad t_p([Q]) = (1+y)/2, \quad t_p([C_r^*D_\infty]) = 1.$$

This shows that the images under  $HS^{C_r^*} : K_0(C_r^*D_\infty) \rightarrow HH_0^{top}(C_r^*D_\infty)$  of the three elements  $[C_r^*D_\infty]$ ,  $[P]$  and  $[Q]$  are  $\mathbb{C}$ -linearly independent in  $HH_0^{top}(C_r^*D_\infty)$ . It follows that  $HS^{C_r^*} : K_0(C_r^*D_\infty) \rightarrow HH_0^{top}(C_r^*D_\infty)$  is injective.  $\square$

### 5. Another result

The following is a partial result on an analogue, in the setting of the maximal  $C^*$ -algebra  $C_{max}^*G$  of a group  $G$ , of the Conjecture stated in the Introduction.

**THEOREM 5.1.** *Let  $G$  be a torsion-free group and  $\phi : G \rightarrow F$  a homomorphism, with  $F$  a finite group. Let  $P$  be a finitely generated projective  $C_{max}^*G$ -module such that  $[P] \in K_0(C_{max}^*G)$  lies in the image of the assembly map*

$$\mu_{C_{max}^*} : K_0(BG) = K_0^G(\underline{E}G) \rightarrow K_0(C_{max}^*G).$$

*Then  $\mathbb{C}F \otimes_\phi P$  is a free  $\mathbb{C}F$ -module of rank  $\epsilon([P]) = \kappa([P])$ .*

**PROOF.** The induced map  $\phi_* : C_{max}^*G \rightarrow C_{max}^*F = \mathbb{C}F$ , composed with the universal trace  $HS^{\mathbb{C}} : \mathbb{C}F \rightarrow HH_0(\mathbb{C}F)$  defines a continuous trace

$$T = HS^{\mathbb{C}} \circ \phi_* : C_{max}^*G \rightarrow HH_0(\mathbb{C}F)$$

and hence induces

$$T : K_0(C_{max}^*G) \rightarrow HH_0(\mathbb{C}F)$$

satisfying  $T([P]) = HS^{\mathbb{C}}(\mathbb{C}F \otimes_\phi P) \in HH_0(\mathbb{C}F)$ . According to Corollary 4.3, this trace value has the form  $\kappa([P]) \cdot HS^{\mathbb{C}}([\mathbb{C}F])$ ; note also that  $\kappa = \epsilon$  on the image of the assembly map  $\mu_{C_{max}^*}$  so  $\epsilon([P]) = \kappa([P])$ . Because  $F$  is finite, two finitely generated projective  $\mathbb{C}F$ -modules  $A$  and  $B$  are isomorphic if and only if  $HS^{\mathbb{C}}([A]) = HS^{\mathbb{C}}([B])$ . Since

$$\kappa([P]) \cdot HS^{\mathbb{C}}([\mathbb{C}F]) = HS^{\mathbb{C}}([\mathbb{C}F]^{\kappa([P])}) = HS^{\mathbb{C}}([\mathbb{C}F \otimes_\phi P])$$

we conclude that  $\mathbb{C}F \otimes_\phi P$  is free, of rank  $\kappa([P]) = \epsilon([P])$ . □

**REMARKS 5.2.** (1) If  $G$  is a torsion-free group which satisfies the Bass Conjecture over  $\mathbb{C}$  and  $M$  is a finitely generated projective  $\mathbb{C}G$ -module, then – by considering the classical Hattori-Stallings trace – the  $\mathbb{C}F$ -module  $\mathbb{C}F \otimes_\phi M$  is free, of rank  $\kappa(M) = \epsilon(M)$ .

(2) The condition in Theorem 5.1 that  $[P]$  lies in the image of the assembly map cannot be dropped: Take a torsion-free infinite group with property (T) and let  $P$  be the projective  $C_{max}^*G$ -module corresponding to the ‘‘Kazhdan Projection’’. Assume that  $G$  admits a homomorphism onto a finite group  $F \neq \{e\}$ . Then  $P$  maps to the trivial representation  $\mathbb{C} = \mathbb{C}F \otimes_\phi P$  of  $F$ , and not to a multiple of the regular representation.

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