Appendix: Twisted Rapid Decay

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Throughout this appendix, Γ is a finitely generated group, endowed with a length function ℓ, and σ is a multiplier on Γ. We adopt the notations used in the first paragraph of the paper.

Definition 0.1. We will say that the group Γ has *σ-twisted Rapid Decay property (with respect to the length ℓ)* if

$$H_\ell^\infty(\Gamma,\sigma) \subseteq C_r^*(\Gamma,\sigma).$$

We just say that the group Γ has the *Rapid Decay property (with respect to the length ℓ)*, if it has the σ-twisted Rapid Decay property (with respect to the length ℓ) for the constant multiplier 1. For short, we shall say that a group Γ has *property σ-RD* if there esists a length function ℓ with respect to which Γ has the σ-twisted Rapid Decay property.

Remark 0.2. In the context of noncommutative geometry, the reduced C^*-algebra $C_r^*(\Gamma,\sigma)$ represents the space of *continuous* functions on a noncommutative manifold, and $H_\ell^\infty(\Gamma,\sigma)$ the space of of *smooth* functions on the same noncommutative manifold. This comes from the abelian case, where using Fourier transforms, one easily sees that $C_r^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ and that $H_\ell^\infty(\mathbb{Z}^n) \cong C^\infty(\mathbb{T}^n)$ (for the word length associated to the generating set $S = \{(\pm 1, 0, \dots), \dots, (0, \dots, \pm 1)\}$ of $\mathbb{Z}^n$). The (σ-twisted) Rapid Decay property can be rephrased as the desirable property that every smooth function on the noncommutative manifold is also a continuous function.

Proposition 0.3. *Let σ be a multiplier on Γ and ℓ be a length function on Γ. The following are equivalent:*

1. *Γ has σ-twisted Rapid Decay (with respect to the length ℓ).*
2. *There exists $C, s > 0$ such that for any $f \in \mathbb{C}(\Gamma,\sigma)$*
$$\|f\|_{op} \leq C\|f\|_s.$$
3. *There exists a polynomial P such that for any $f \in \mathbb{C}(\Gamma,\sigma)$ and f supported in a ball of radius r*
$$\|f\|_{op} \leq P(r)\|f\|_{\ell^2\Gamma}.$$
4. *There exists a polynomial P such that for any $f, g \in \mathbb{C}(\Gamma,\sigma)$ and f supported in a ball of radius r*
$$\|f *_\sigma g\|_{\ell^2\Gamma} \leq P(r)\|f\|_{\ell^2\Gamma}\|g\|_{\ell^2\Gamma}.$$

Proof. (1) $\Leftrightarrow$ (2) As in the case of untwisted Rapid Decay, the inclusion $H_\ell^\infty(\Gamma,\sigma) \subseteq C_r^*(\Gamma,\sigma)$ is continuous since both inclusions $H_\ell^\infty(\Gamma,\sigma) \subseteq \ell^2\Gamma$ and $C_r^*(\Gamma,\sigma) \subseteq \ell^2\Gamma$ are continuous. Since $H_\ell^\infty(\Gamma,\sigma)$ is a Fréchet space, the continuity of the inclusion $H_\ell^\infty(\Gamma,\sigma) \subseteq C_r^*(\Gamma,\sigma)$ rephrases as the statement of (2). The converse is obvious since $H_\ell^{s+1}(\Gamma) \subseteq H_\ell^s(\Gamma)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Take $f \in \mathbb{C}(\Gamma,\sigma)$ supported in a ball of radius r, then

$$\|f\|_{op} \leq C\|f\|_s = C\sqrt{\sum_{\gamma\in\Gamma} |f(\gamma)|^2(1+\ell(\gamma))^{2s}} \leq C(1+r)^s\|f\|_{\ell^2\Gamma}.$$

Hence (3) follows. Since $\|f\|_{op} = \sup\{\frac{\|f*_\sigma g\|_{\ell^2\Gamma}}{\|g\|_{\ell^2\Gamma}}|0 \neq g \in \ell^2\Gamma\}$ we deduce (4) as well. That (4) implies (3) is by definition of the operator norm.

(3) $\Rightarrow$ (2) For $n \in \mathbb{N}$, denote by $S_n = \{\gamma \in \Gamma | n \leq \ell(\gamma) < n+1\}$ the sphere of radius n. For $f \in \mathbb{C}(\Gamma, \sigma)$ we have:

$$\|f\|_{op} = \|\sum_{n=0}^{\infty} \lambda_\sigma(f|_{S_n})\|_{op} \leq \sum_{n=0}^{\infty} \|f|_{S_n}\|_{op},$$

so that using (3) we get the following bound

$$\begin{aligned} \|f\|_{op} &\leq \sum_{n=0}^{\infty} P(n+1)\|f|_{S_n}\|_{\ell^2\Gamma} \leq \sum_{n=0}^{\infty} C(n+1)^k \ |f|_{S_n}\|_{\ell^2\Gamma} \\ &\leq C\sqrt{\sum_{n=0}^{\infty}(n+1)^{-2}}\sqrt{\sum_{n=0}^{\infty}(n+1)^{2k+2}\|f|_{S_n}\|^2_{\ell^2\Gamma}} \leq C'\|f\|_{k+1} \end{aligned}$$

where C' is some constant bigger than $C\pi/6$. □

The following proposition was known by Ji and Schweitzer [JiSc], but the proof we give here might be shorter.

Lemma 0.4. *Let ℓ be a length function on Γ, if Γ has Rapid Decay (with respect to the length ℓ), then Γ has σ-twisted Rapid Decay (with respect to the length ℓ) for any multiplier σ.*

Proof. Take $\gamma \in \Gamma$, then:

$$|f *_\sigma g(\gamma)| = |\sum_{\mu\in\Gamma} f(\gamma^{-1}\mu)g(\mu)\sigma(\gamma^{-1}\mu,\mu)| \leq \sum_{\mu\in\Gamma} |f(\gamma^{-1}\mu)|\,|g(\mu)| = |f| * |g|(\gamma)$$

so that summing and squaring over $\gamma \in \Gamma$ yields

$$\|f *_\sigma g\|_{\ell^2\Gamma} \leq \||f| * |g|\|_{\ell^2\Gamma} \leq P(r)\|f\|_{\ell^2\Gamma}\|g\|_{\ell^2\Gamma}$$

and we conclude that Γ has σ-twisted Rapid Decay using the previous proposition. □

The following corollary is the first part of Proposition 2.1 in [La2] with an obvious modification.

Corollary 0.5 (Noncommutative Sobolev Embedding Theorem)**.** *Let ℓ be a length function on Γ, if Γ has Rapid Decay (with respect to the length ℓ), then there is S sufficiently large such that for any multiplier σ on Γ and any $s \geq S$, $H^s_\ell(\Gamma,\sigma)$ is a Banach algebra such that $H^s_\ell(\Gamma,\sigma) \subseteq C^*_r(\Gamma,\sigma)$.*

Proof. Let s be bigger than the degree of the polynomial of point (3) in Proposition 0.3. We first have to show that there is a constant $K = K(s)$ such that for any $f, g \in \mathbb{C}(\Gamma,\sigma)$, then $\|f *_\sigma g\|_s \leq K\|f\|_s\|g\|_s$. But this is true since $\|f *_\sigma g\|_s \leq \||f| * |g|\|_s$ and $\||f| * |g|\|_s \leq K'\|f\|_s\|g\|_s$ by Proposition 2.1 part (a) in [La2] (see also Proposition 8.15 in [Va]) since we assumed that Γ has Rapid Decay (with respect to the length ℓ). Therefore $H^s_\ell(\Gamma,\sigma)$ is a Banach algebra. By Lemma 0.4, we know that since Γ has property RD, then Γ has property σ-twisted RD for any multiplier σ on Γ, and hence $H^s_\ell(\Gamma,\sigma) \subseteq C^*_r(\Gamma,\sigma)$ follows from Proposition 0.3 part (2). □

Remark 0.6. In the context of noncommutative geometry, Corollary 0.5 can be viewed as the analog of the Sobolev Embedding Theorem for a compact manifold M, a simplified version of which saying that any function in the Sobolev space $W^{s,2}(M)$ for $s > \dim M/2$ is actually continuous. Indeed, using Fourier transforms, on can see that $W^{s,2}(\mathbb{T}^n) \simeq H^s_\ell(\mathbb{Z}^n)$ for the word length associated to the generating set $S = \{(\pm 1, 0, \dots), \dots, (0, \dots, \pm 1)\}$ of $\mathbb{Z}^n$, and that $C^*_r(\mathbb{Z}) \simeq C(\mathbb{T}^n)$.

Example 0.7. Groups having Rapid Decay notably include: Polynomial growth groups (Jolissaint [Jo]), free groups (Haagerup [Ha]) and more generally Gromov hyperbolic groups (Jolissaint-de la Harpe [dH]), cocompact lattices in $SL_3(F)$ where F is the p-adic field $\mathbb{Q}_p$, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $E_{6(-26)}$, as well as finite products of rank one Lie groups (see Rammagge-Robertson-Steger [RaRoSt], Lafforgue [La2] and [Ch]) and all lattices in a rank one Lie group, see [ChR].

Question: Is it possible to find a group Γ which doesn't have Rapid Decay, but which has σ-twisted Rapid Decay for some multiplier σ on Γ (or does the converse of Lemma 0.4 hold)?

The following is the second part of Proposition 1.2 of [La2] with a trivial change. But we still recall Lafforgue's proof below for the sake of completeness.

Proposition 0.8. *Let ℓ be a length function on Γ, if Γ has Rapid Decay (with respect to the length ℓ), then for any multiplier σ on Γ and for s sufficiently large (and also for $s = \infty$), the inclusion $H^s_\ell(\Gamma, \sigma) \hookrightarrow C^*_r(\Gamma, \sigma)$ induces an isomorphism in K-theory.*

Proof. The idea of the proof is as follows. By Corollary 0.5, there exists $S > 0$ and finite such that for any $s \geq S$, then $H^s_\ell(\Gamma, \sigma) \subseteq C^*_r(\Gamma, \sigma)$, and since $\mathbb{C}(\Gamma, \sigma) \subseteq H^s_\ell(\Gamma, \sigma)$, it follows that $H^s_\ell(\Gamma, \sigma)$ is a dense $*$-subalgebra of $C^*_r(\Gamma, \sigma)$. All we have to show is that the inclusion $H^s_\ell(\Gamma, \sigma) \subseteq C^*_r(\Gamma, \sigma)$ is spectral, it then follows (see e.g. Proposition 8.14 of [Va]) that the inclusion $H^s_\ell(\Gamma, \sigma) \hookrightarrow C^*_r(\Gamma, \sigma)$ induces an isomorphism in K-theory.

Now, for two number s, t such that $S < t < s$ the first step is to show that $H^s_\ell(\Gamma, \sigma)$ is stable by holomorphic functional calculus in $H^t_\ell(\Gamma, \sigma)$. To do so, and since $H^s_\ell(\Gamma, \sigma)$ is dense in $H^t_\ell(\Gamma, \sigma)$, it is enough (see Remark 8.13 in [Va]) to prove that the spectral radius $\rho_s(f)$ of $f \in H^s_\ell(\Gamma, \sigma)$ is the same as $\rho_t(f)$, the one of $f \in H^t_\ell(\Gamma, \sigma)$, namely that

$$\lim_{n \to \infty} \|f^{*_\sigma n}\|_s^{1/n} = \lim_{n \to \infty} \|f^{*_\sigma n}\|_t^{1/n}, \tag{0.1}$$

where for $n \in \mathbb{N}$ we set $f^{*_\sigma n} = \underbrace{f *_\sigma f *_\sigma \cdots *_\sigma f}_{n}$. Notice that since $t < s$, then $\| \, \|_t \leq \| \, \|_s$ and hence $\rho_t(f) \leq \rho_s(f)$, so we only need to prove the other inequality. For $\gamma \in \Gamma$, using the triangle inequality one sees that

$$\begin{aligned} |f^{*_\sigma n}(\gamma)| &\leq \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} |f(\gamma\gamma_1^{-1})||f(\gamma_1\gamma_2^{-1})| \dots |f(\gamma_{n-2}\gamma_{n-1}^{-1})||f(\gamma_{n-1})| \\ &= \sum_{\gamma_1 \dots \gamma_n = \gamma} |f(\gamma_1)| \dots |f(\gamma_n)| \end{aligned}$$

Therefore, using that $(1+\ell(\gamma))^{s-t} \leq n^{s-t} \sum_{i=1}^n (1+\ell(\gamma_i))^{s-t}$ if $\gamma_1 \dots \gamma_n = \gamma$ (which follows easily from Lemma 1.1.4 (3) in [Jo]) we deduce that

$$\|f^{*_\sigma n}\|_s = \|(1+\ell)^{s-t} f^{*_\sigma n}\|_t \leq n^{s-t+1} K^{n-1} \|f\|_s \|f\|_t^{n-1},$$

where $K = K(t)$ is the constant in the proof of Corollary 0.5. Taking the n-th root and the limit shows that $\lim_{n\to\infty} \|f^{*_\sigma n}\|_s^{1/n} \le K\|f\|_t$. Replacing f by $f^{*_\sigma m}$ in the previous inequlity, taking the m-th root and the limit shows $\rho_s(f) \le \rho_t(f)$. We can now show that $H^s_\ell(\Gamma,\sigma) \subseteq C^*_r(\Gamma,\sigma)$ is spectral, namely that for $f \in H^s_\ell(\Gamma,\sigma)$, then its spectral radius $\rho_s(f)$ equals $\rho_*(f)$, its spectral radius as an element of $C^*_r(\Gamma,\sigma)$. If $\rho_s(f) = 0$, it is clear because $\rho_*(f) \le \rho_s(f)$. Otherwise, Hölder's inequality shows that

$$\|f\|_t \le \|f\|_s^{\frac{t}{s}} \|f\|_{\ell^2\Gamma}^{\frac{1-t}{s}},$$

and hence

$$|f^{*_\sigma n}\|_{op} \ge \|f^{*_\sigma n}\|_{\ell^2\Gamma} \ge \|f^{*_\sigma n}\|_t^{\frac{s}{s-t}} \|f^{*_\sigma n}\|_s^{-\frac{t}{s-t}},$$

so that we conclude using equality (0.1). □

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